Classical Rings

by

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Abstract

Classical rings are rings in which every element is zero, a zero divisor, or a unit. In this study, we present properties which allow us to determine whether or not a ring is classical. We begin our search with finite rings and conclude that all finite rings are classical. We study formal fractions both in commutative and noncommutative rings, determining that the total rings of quotients when they exist are classical. For more general cases in noncommutative rings, we find that matrix rings are classical if and only if the corresponding ring from which the entries of the matrix originate was classical. We end our study with a look at chain conditions, concluding that Artinian rings are classical and, more generally, that rings of Krull dimension 0 are classical.

1 Introduction

A classical ring is a ring in which every element is either zero, a unit, or a zero divisor. Such a definition is quite simple, and many early students of algebra may believe that all rings are of this form. Clearly, this is not the case, but a classical ring is still an interesting structure deserving of discussion. The structure of a classical ring is what many would call "nice", but unlike many other types of rings, we lack a clear internal characterization that allows us to always determine whether a ring is classical or not.

In this paper we begin our study of classical rings by laying out necessary background information, including definitions that will be relevant. We first define structures in algebra such as a ring in order to prepare the reader with necessary terminology that will be used throughout. We define a zero divisor, unit, and identity element as just a few of the necessary types of ring elements, and then proceed to define structural attributes of rings.

Our quest to understand when a ring is classical begins with the study of finite rings. We present theorems concerning left and right multiplication operators in finite rings. Such theorems include that left multiplication by an element is surjective in a ring with unity if and only if that element is a left unit, and that surjective left (or right) multiplication implies that right (or left) multiplication is injective in a ring with unity. We proceed to use the power of the finite condition on the rings in order to prove that any element in a finite ring must be a unit, a zero divisor, or zero, and so we have that all finite rings must be classical.

We next continue with a discussion of the history of the ring of quotients as well as present Grell's commutative ring of quotients, which we carefully construct. Upon completing this construction, we understand that a commutative ring is classical if and only if that ring is isomorphic to its total ring of quotients.

Our journey continues into the realm of noncommutative rings. In this section, we begin by discussing the construction of the noncommutative ring of quotients. This construction is different from the commutative ring of quotients and involves the invocation of what is known as the Ore condition in order to embed a noncommutative ring as with the commutative case. After completing this construction, we then find that the noncommutative ring of quotients is also classical. Of course, the noncommutative ring of quotients does not describe every type of noncommutative ring. We then present an example of a classical ring which is noncommutative in which elements are able to behave as a left zero divisor and a left unit. This example presents us with a strange case in which a single element can behave as both a unit and a zero divisor. Finally, we explore the noncommutative matrix rings in an attempt to classify when matrix rings will be classical. Through the use of McCoy's theorem, we are able to determine if a matrix ring is classical. Interestingly, we are also able to find an interesting result that a commutative ring with unity is classical if the corresponding noncommutative ring of square matrices is also classical.

Our journey ends in the collection of rings with chain conditions. Such rings include the Artinian and Noetherian rings. We initially prove that all Artinian commutative rings are classical, and then explain why the same is not true of Noetherian rings. Our discussion of Artinian and Noetherian rings lead us to a discussion of Krull dimension, and we find that all Artinian rings are in fact Noetherian rings which have Krull dimension zero. This then tells us that there is a strong link between Krull dimension 0 and classical.

2 Definitions and Background

The following background material not only lays out relevant definitions and theorems that will help the reader to better understand the topic of this study, but also includes important notation. We shall begin by defining a group and similar structures in order to give the reader a basis for future discussion.

Definition 2.1. *Let G be a set closed under a binary operation, denoted by $*$. Then G is called a **group** if it satisfies the following axioms:*

1. *Associativity: $(a * b) * c = a * (b * c)$, for all a, b, $c \in G$.*
2. *Identity: There exists an element $e \in G$ such that for all $a \in G$ we have that $a * e = e * a = a$. (Note that due to this axiom, we have that a group is nonempty.)*
3. *Inverses: For each $a \in G$ there is an element $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1}a$. We call a^{-1} the inverse of a.*

*Additionally, G is an **abelian group** if for all a, $b \in R$, $a * b = b * a$.*

We sometimes find in necessary when working with structures such as groups to define weaker structures as well. We shall define two, the semigroup and monoid. A **semigroup** is a set which is closed under an associative binary operation. If the operation is also commutative, then we have a **commutative semigroup**. A **monoid** is a semigroup that has an identity element for its binary operation. Adding on the condition for commutativity, we have a **commutative monoid**.

Groups are not the only type of structure we desire to look at. We shall now define a ring, and the different types of rings that are encountered in this study. Also included will be relevant definitions in order to more perfectly explain the types of objects within and relating to these rings.

Definition 2.2. *Let R be a set on which two binary operations are defined, called addition and multiplication, and denoted by $+$ and juxtaposition respectively. Then R is called a **ring** with respect to these operations if the following properties hold:*

1. *Closure: If $a, b \in R$, then the sum $a + b$ and the product ab are defined and belong to R.*

2. *Associative Laws: For all $a, b, c \in R$, $a + (b + c) = (a + b) + c$ for addition and $a(bc) = (ab)c$ for multiplication.*

3. *Additive Identity: The set contains an additive identity element, denoted by 0, such that for all $a \in R$, $a + 0 = a = 0 + a$.*

4. *Additive Inverses: For all $a \in R$, there exists an $x \in R$ such that $a + x = 0$ and $x + a = 0$. This x is called the additive inverse of a, and is denoted $-a$.*

5. *Commutativity: For all $a, b \in R$, $a + b = b + a$.*

6. *Distributive laws: For all $a, b, c \in R$, $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.*

This means that R is an abelian group under addition and a semigroup under multiplication where the distributive laws tie these operations together.

*Furthermore, we call R a **commutative ring** if for all $a, b \in R$ $ab = ba$. Note: A ring with the absence of a multiplicative identity is sometimes called a **rng**.*

Additionally, we define a ring that contains a multiplicative identity separately.

Definition 2.3. *A ring R is called a commutative ring with unity (or a ring with 1) if there exists $1 \in R$ such that for all $a \in R$, $1a = a = a1$.*

Definition 2.4. *Additionally, noncommutative rings may contain what we shall refer to as a **left identity or right identity**. A left identity element is an element 1_L such that for all $x \in R$, $1_L a = a$. A right identity element is an element 1_R such that for all $x \in R$, $a 1_R = a$.*

Example 2.5. *Examples of rings include $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rational numbers), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers). These rings are all commutative rings. For an example of a noncommutative ring, consider $M_n(R) = R^{n \times n}$, the colloection of all $n \times n$ matrices having elements of R as entries, where R is any ring.*

Additionally, we shall define a subring and subring test. Those familiar with group theory will recognize the following definition as a ring analogue of a subgroup.

Definition 2.6. *Let R be a ring. A nonempty subset S of R is called a **subring** of R if it is a ring under the addition and multiplication of R.*

In order to characterize a set as being a subring, we often use what is called the subring test. The criteria for the test are equivalent to the definition of a subring and give a more efficient method of checking whether or not a subset is a subring.

Theorem 2.7. *(Subring Test) A nonempty subset S of R is a subring of R if and only if x,$y \in R$ implies that $x - y \in R$ and $xy \in R$.*

Identity elements are not the only special elements that can show up in a ring. There are also elements such as units.

Definition 2.8. *Let R be a ring with unity. If $a \in R$ and there exists $x \in R$ such that $ax = 1$, then a is a left unit with right inverse x. If $xa = 1$, then a is a right unit with left inverse x. If a is both a left and right unit, then a is a unit.*

Remark 2.9. *If a is both a left and right unit, then its left and right inverses coincide and it has a unique inverse, denoted a^{-1}.*

Proof: *Let $ab = 1$ and $ca = 1$ for $a, b, c \in R$. Then $c = c1 = c(ab) = (ca)b = 1b = b$.*

An interesting example occurs in 3.4, in which we find that there are multiple inverses.

Definition 2.10. *Let R be a ring with unity. We call the set of all units in R the group of units, and it is denoted by $R^\times = U(R)$.*

We shall now prove that $R^\times$ is a group under multiplication.

Proof: We begin by noting that the identity is a unit, and is its own inverse. Thus, our set of units is nonempty and contains the identity.

To check closure, we must see that the product of two units is a unit. Let a, $b \in R^\times$. Now, since a and b are units, we know there exists a^{-1}, $b^{-1} \in R$ such that $aa^{-1} = a^{-1}a = 1$ and $bb^{-1} = b^{-1}b = 1$. Notice that this means $(a^{-1})^{-1} = a$ and $(b^{-1})^{-1} = b$ thus $a^{-1}, b^{-1} \in R^\times$. Notice that $(ab)(b^{-1}a^{-1}) = a(1)a^{-1} = 1$ and likewise $(b^{-1}a^{-1})(ab) = 1$. So the product of units gives a unit and in fact $(ab)^{-1} = b^{-1}a^{-1}$.

For associativity, recall that our elements in our set of units were originally in R. Therefore, associativity is inherited from the ring R. Thus, the group axioms are met and we have that $R^\times$ is a group. ■

Consider $\mathbb{Z}$. In this ring, the only units are 1 and -1, so $\mathbb{Z}^\times = \{\pm 1\}$. In $\mathbb{Q}$, every nonzero element is a unit, so $\mathbb{Q}^\times = \mathbb{Q} - \{0\}$.

Example 2.11. *Consider $\mathbb{Z}_{14}$. In this ring, 1 and $-1 = 13$ are units. Note that $3 \cdot 5 = 1$, so we can see that 3 and 5 are units. Since they are units, $-3 = 11$ and $-5 = 9$ will also be units. In general, the units of $\mathbb{Z}_n$ are determined by the integers relatively prime to n. So for $\mathbb{Z}_{14}$, we find that only 1, 3, 5, 9, 11, and 13 are units in this ring. In other words, $(\mathbb{Z}_{14})^\times = U(14) = \{1, 3, 5, 9, 11, 13\}$.*

A discussion of units necessarily leads to a definition for fields and other similar rings.

Definition 2.12. *Let R be a ring with $1 \neq 0$. If every nonzero element of R is a unit, $R^\times = R - \{0\}$, then R is a **division ring** (or **skew field**). A **field** is a commutative division ring. A noncommutative division ring is called a "**strictly skew field**."*

Using the above definition, $\mathbb{Z}$ is not a field, as $2^{-1} = 1/2 \notin \mathbb{Z}$, and therefore it is not a unit. $\mathbb{Q}$ and $\mathbb{R}$ are fields, as every nonzero element in each is a unit. Now, we shall give an example of a noncommutative, or skew, field.

Example 2.13. *Consider $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$. We call this ring the Hamiltonians, or the quaternions. This ring is a strictly skew field, as it has inverses for all nonzero elements, but it is not commutative, and therefore not a field.*

In the quaternions, we define multiplication such that $i^2 = -1$, $j^2 = -1$, and $k^2 = -1$. Also, $ij = k$, $jk = i$, and $ki = j$. It is important to note however that we define $ji = -k$,

$kj = -i$, and $ik = -j$. With multiplication among units defined, we extend linearly to all quaternions.

Because of this definition for multiplication, one can quickly see that the quaternions are not commutative. We will now quickly show that the quaternions do in fact have inverses. Let $z = a + bi + cj + dk$ and define $\bar{z} = a - bi - cj - dk$ and so $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2 + c^2 + d^2}$. Because $\overline{wz} = \bar{w} \cdot \bar{z}$ and since $|z| \neq 0$ when $z \neq 0$, $z \cdot \frac{\bar{z}}{|z|^2} = \frac{z\bar{z}}{z\bar{z}} = 1$ and $\frac{\bar{z}}{|z|^2} \cdot z = \frac{\bar{z}z}{|z|^2} = 1$ and so $z^{-1} = \frac{\bar{z}}{|z|^2}$ (inverses for nonzero elements exist).

Units are not the only special types of elements we shall encounter. In addition to units, there are also elements known as idempotents, nilpotents, and zero divisors. While zero divisors are commonly discussed elements in rings, idempotents and nilpotents are less common. We shall define these elements here as they will be important later.

An element $x \in R$ is said to be **idempotent** if $x^2 = x$. Note that the identity element is idempotent in any ring. An element $x \in R$ is said to be **nilpotent** if there exists a positive integer n such that $x^n = 0$. For an example of a nilpotent element, consider 2 in the ring $\mathbb{Z}_8$. We see that $2^3 = 0$, so 2 is nilpotent in this ring.

Definition 2.14. *If a and b are two nonzero elements of a ring R such that $ab = 0$, then a and b are* **zero divisors***. Specifically, a is a* **left zero divisor** *and b is a* **right zero divisor**.

For an example of zero divisors, consider $\mathbb{Z}_6$. Note that $2 \cdot 3 = 0$ in $\mathbb{Z}_6$ but $2, 3 \neq 0$. Therefore, 2 and 3 are zero divisors.

The absence of zero divisors in a ring results in the conclusion that if a product is zero, then one of the two factors must be zero. Since we have that one of the factors must be zero, we can then motivate the use of the cancellation laws for multiplication. Before we prove this, we shall introduce the concept of a homomorphism as well as a relevant theorem.

Definition 2.15. *Let R and S be rings. A map $\phi : R \mapsto S$ is called a* **ring homomorphism** *if for all $a, b \in R$,*

- $\phi(a+b) = \phi(a) + \phi(b)$
- $\phi(ab) = \phi(a)\phi(b)$

*If our ring homomorphism is bijective, then we have that it is an **isomorphism**. Note that there are other types of morphisms on rings. One can have a **monomorphism**, which is a one-to-one homomorphism, an **endomorphism**, which is a homomorphism from a ring to itself, and an **epimorphism**, which is an onto homomorhpism. An **automorphism** is an isomorphism from a ring to itself.*

When working exclusively with rings with unity, one also requires that $\phi(1) = 1$ (multiplicative identity of R is sent to the multiplicative identity of S).

Example 2.16. *Let F be the ring of all functions that map $\mathbb{R}$ into $\mathbb{R}$. For each $a \in \mathbb{R}$, we have the **evaluation homomorphism** $\phi_a : F \to \mathbb{R}$ given by $\phi_a(f) = f(a)$, where $f \in F$.*

When discussing homomorphisms, we must also discuss the kernel of the homomorphism.

Definition 2.17. *Let $\phi : R \mapsto S$ be a homomorphism. The set*

$$\ker(\varphi) = \{a \in R \mid \phi(a) = 0\}$$

*is called the **kernel** of ϕ.*

For any homomorphism, the kernel of the homomorphism is simply anything that is mapped to 0.

Example 2.18. *The map ϕ from $\mathbb{C}$ to the ring of 2×2 real matrices, $\phi : \mathbb{C} \to \mathbb{R}^{2\times 2}$, given by $\phi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a ring monomorphism. If we restrict the codomain down to matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, then ϕ becomes an isomorphism.*

The knowledge of the kernel can also lead to a powerful theorem which is known as the First Isomorphism Theorem, which gives a result concerning the relationship between a ring acting as the domain of an isomorphism, its kernel, and the image of the isomorphism.

Theorem 2.19 (First Isomorphism Theorem)**.** *Let R and S be rings, and $\phi : R \mapsto S$ be a ring homomorphism. Then the image of ϕ is isomorphic to $R/\ker(\phi)$. If ϕ is surjective, then $R/\ker(\phi) \cong S$.*

We will not prove the First Isomorphism Theorem, as that is not relevant to our discussion. Instead, we shall now prove some important theorems involving an injective homomorphism and kernel. From there we shall proceed to show how the absence of zero divisors in a ring is equivalent to having the left and right cancellation laws.

Theorem 2.20. *A homomorphism ϕ is injective if and only if its kernel is trivial (the subset $\{0\}$ of R).*

Theorem 2.20 will be a valuable conclusion that we will use in the proof of our theorem concerning cancellation laws in an integral domain. As such, we will give a short sketch of a proof for this theorem. If we have a homomorphism ϕ that is injective, necessarily only one element in the domain may be sent to 0. By our conditions on homomorphism, this element must be 0. Conversely, if the kernel is trivial, then we can take two equal images of elements and through the homomorphism property (preserving addition) conclude that their difference maps to 0. Thus their difference lies in the kernel and so their difference is 0 (i.e. they are equal). Also, note that we could have taken two equal images of elements through the homomorphism property that preserves multiplication and show that group homomorphisms are also injective.

We shall define left and right multiplication operators (for some $a \in R$ where R is a ring) as follows:

- Left multiplication by a will be denoted as $\mathcal{L}_a(x) = ax$.
- Right multiplication by a will be denoted as $\mathcal{R}_a(x) = xa$.

Notice that the distributive laws say that $\mathcal{L}_a(x + y) = a(x + y) = ax + ay = \mathcal{L}_a(x) + \mathcal{L}_a(y)$ which is exactly that they're group homomorphisms under addition.

Theorem 2.21. *Let R be a ring, possibly without unity, and $0 \neq a \in R$. $\mathcal{L}_a$ is injective if and only if a is not a left zero divisor. Likewise, $\mathcal{R}_a$ is injective if and only if a is not a right zero divisor. Both $\mathcal{L}_a$ and $\mathcal{R}_a$ are injective if and only if a is not a zero divisor.*

Proof: Given a ring, R, possibly without unity, a, $x \in R$, note that $\mathcal{L}_a(x) = ax$ is a group homomorphism (preserving addition). Now, if we have that $\mathcal{L}_a$ is injective, then we have that the kernel of $\mathcal{L}_a$ is trivial, as we know that a group homomorphism is injective if and only if the kernel is trivial. Now, recall that the kernel of a homomorphism is everything that maps to 0. Thus, $ax = 0$ if and only if $x = 0$, since the kernel is trivial. If this is true, we have proven that a is not a left zero divisor.

Suppose a is not a left zero divisor. Then, $ax = 0$ if and only if $x = 0$. This implies that the kernel is trivial, and since the kernel is trivial, $\mathcal{L}_a$ must be injective.

Similarly, the argument can be made that right multiplication by a, denoted $\mathcal{R}_a(x) = xa$, is injective if and only if a is not a right zero divisor. $\blacksquare$

Theorem 2.22. *The multiplicative cancellation laws hold in a ring R if and only if R has no zero divisors. Specifically, let $a, b, c \in R$. If a is not a left zero divisor and not zero, then $ab = ac$ implies that $b = c$. If a is not a right zero divisor and not zero, then $ba = ca$ implies that $b = c$.*

This theorem is a direct result of the previous theorem. Because left and right multiplication for a nonzero element are injective if and only if the element is not a zero divisor, we find that cancellation laws hold as long as there are no zero divisors in our ring.

Like fields are commutative rings in which every nonzero element is a unit, we define a name for a commutative ring with no zero divisors.

Definition 2.23. *Let R be a ring with unity $1 \neq 0$ containing no zero divisors. Then R*

is a domain. We call a commutative domain an integral domain.

Note: Some authors will use the above interchangeably. In particular, Cohn uses integral domain when referring to a noncommutative ring [C].

Example 2.24. *$\mathbb{Z}$ is an integral domain, and $\mathbb{Z}_p$ is as well (for any prime p). Note that $\mathbb{Z}_n$ for any n that is not prime is not an integral domain, as in the case of $\mathbb{Z}_6$.*

Upon looking at a comparison between fields and integral domains, one can make the following conclusions about the two.

Theorem 2.25. *Every skew field is a domain. In particular, every field is an integral domain.*

Proof: Theorem 3.1 states that being a unit is the same as having surjective left and right multiplication maps. Theorem 3.2 says that if the maps are surjective, then they are injective. Therefore, the multiplication maps (for nonzero elements) are all injective. Thus there are no zero divisors. ■

Theorem 2.26. *Every finite domain is a skew field. In particular, every finite integral domain is a field.*

Proof: A map from a finite set to itself is injective if and only if it is surjective. So if all nonzero elements are not zero divisors, their multiplication maps are injective. This implies that they are surjective and so by Theorem 3.1 they are units. ■

Remark 2.27. *Wedderburn in 1905 proved that every finite skew field is a field [M-W]. For a proof of this theorem, see I. N. Herstein's* Noncommutative Rings *[H]. This means that for finite rings the following are equivalent: domain, integral domain, skew field, and field.*

We shall now define the characteristic of a ring with unity.

Definition 2.28. *The **characteristic** of a ring with unity R, denoted $\mathrm{char}(R)$, is the smallest positive integer n such that $n1 = \underbrace{1+1+\cdots+1}_{n-\text{times}} = 0$. Note that if no such integer exists, then we say that the ring is of characteristic 0.*

Example 2.29. *$\mathbb{Z}_n$ is a ring of characteristic n. $\mathbb{C}$ is a ring of characteristic 0.*

As we have now introduced the many types of elements that can be found in a ring, we shall define a particular type of ring which we have chosen to look at with this study.

Definition 2.30. *We define a ring as being **left classical** if every element in the ring is either zero, a left zero divisor, or a left unit relative to some left identity. Likewise, we define a ring as **right classical** if every element in the ring is either zero, a right zero divisor, or a right unit relative to some right identity. If a ring is both left and right classical, then we say the ring is **classical**, as in the case in which R has unity.*

Though this type of ring has long been studied, it was T. Y. Lam who coined the term *classical ring* ([L] page 320).

Example 2.31. *General examples of classical rings include all fields, including $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. $\mathbb{Z}$ is not a classical ring, as 2 is not zero, a unit, or zero divisor in this ring.*

Example 2.32. *A **Boolean ring** is a ring R with unity such that every element in R is idempotent ($x^2 = x$ for all $x \in R$). It turns out that every Boolean ring is classical. Why? Let R be a Boolean ring, so every element of R is an idempotent. That is, $x^2 = x$ for all $x \in R$. Thus, $x^2 - x = 0$, and so $x(1 - x) = 0$. Therefore, either $x = 1$ (i.e. x is a unit), $x = 0$, or $x \neq 0$, $x - 1 \neq 0$ so x is a zero divisor. Therefore R is classical.*

Classical rings are a very "nice" type of ring that can be confusing to introductory students, simply because they are the type of ring that a student would hope every ring would look like. These rings are very special, and tend to show up in interesting places in ring theory. Before we begin making conclusions about this type of ring, we shall proceed to discuss a few more topics of interest that will show up in our paper.

Definition 2.33. *Let R be a ring. An additive subgroup I of R is an **(two-sided) ideal** if it satisfies the following properties for all $a \in I$ and $r \in R$, we have both $ra \in I$ (i.e. $RI \subseteq I$) and $ar \in I$ (i.e. $IR \subseteq I$). We denote an ideal I of R as $I \lhd R$.*

*We call these properties absorption on the left and right, respectively. If an additive subgroup only absorbs multiplication on the right, then we call it a **right ideal**. Similarly, if an additive subgroup only absorbs multiplication on the left, then we call it a **left ideal**.*

Those familiar with group theory will likely recognize the ideal as being the ring theoretic analogue of the normal subgroup. Ideals are valuable tools that we can use to more perfectly understand the structure of a ring.

Example 2.34. *It is not difficult to see that $n\mathbb{Z} = (n) = \{nm \mid m \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$. It is clearly a subring, and for all $x \in \mathbb{Z}$, $x(nm) = (nm)x = n(mx) \in \mathbb{Z}$. Thus, we have* *absorption on the left and right, and therefore we conclude that $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$. In fact, these are the only ideals of $\mathbb{Z}$.*

Remark 2.35. *Every nonzero ring R has at least two ideals, the **improper ideal** R and the **trivial ideal** $\{0\}$. If R is not the zero ring and these are the only ideal, R is called a **simple** ring.*

We shall now introduce a few theorems concerning ideals in order to introduce vocabulary as well as introduce the reader to a few of the more interesting concepts that we can discuss through the use of ideals.

Theorem 2.36. *If R is a ring with unity, and I is an ideal of R containing a unit, then $I = R$.*

Definition 2.37. *A **maximal ideal** of a ring R is an ideal $M \neq R$ such that there exists no proper ideal I of R containing M.*

In $\mathbb{Z}$, we find that the maximal ideals are those ideals that are generated by prime numbers. Furthermore, if we look at a field, we find that the only maximal ideal is $\{0\}$.

Corollary 2.38. *A field contains no proper nontrivial ideals.*

The previously mentioned fact that the $\{0\}$ ideal was maximal in a field can be easily seen to be a result of the above corollary.

Definition 2.39. *An ideal $I \neq R$ in a commutative ring R is a **prime ideal** if $ab \in I$ for some $a, b \in R$ implies that either $a \in I$ or $b \in I$.*

Example 2.40. *$\{0\}$ is a prime ideal in $\mathbb{Z}$. In fact, for a commutative ring with $1 \neq 0$, $\{0\}$ is a prime ideal if and only if R is an integral domain.*

Theorem 2.41. *Let R be a commutative ring with unity. Then M is a maximal ideal of R if and only if R/M is a field.*

Theorem 2.42. *Let R be a commutative ring with unity. Then P is a prime ideal of R if and only if R/P is an integral domain.*

Corollary 2.43. *Every maximal ideal in a commutative ring R with unity is a prime ideal.*

Definition 2.44. *If R is a commutative ring with unity and $a \in R$, the ideal $\{ra \mid r \in R\}$ is the **principal ideal generated by** a and is denoted by (a). An ideal I of R is a **principal ideal** if $I = (a)$ for some $a \in R$. An integral domain in which all ideals are principal is called a **principal ideal domain (PID)**.*

Example 2.45. *We have already noted that every ideal in $\mathbb{Z}$ is principal, so $\mathbb{Z}$ is a PID.*

Definition 2.46. *A partially ordered set has the **ascending chain condition (ACC)** if every strictly ascending sequence of elements eventually terminates. That is, given the sequence $a_1 \leq a_2 \leq \cdots \leq a_r \leq \cdots$, there exists some positive integer n such that $a_n = a_{n+1} = a_{n+2} = \cdots$. In words, every ascending chain eventually stabilizes.*

Definition 2.47. *A **Noetherian ring** is a ring that satisfies the ascending chain condition on ideals. If a ring satisfies the ascending chain condition on left ideals, then it is **left Noetherian**. If a ring satisfies the ascending chain condition on right ideals, then it is **right Noetherian**.*

Example 2.48. *$\mathbb{Z}$ is a Noetherian ring, as any ascending chain of ideals must eventually stabilize. For example, $12\mathbb{Z} \subsetneq 6\mathbb{Z} \subsetneq 2\mathbb{Z} \subsetneq \mathbb{Z}$.*

Definition 2.49. *A partially ordered set has the* ***descending chain condition (DCC)*** *if every strictly descending sequence of elements eventually terminates. That is, the sequence* $a_1 \geq a_2 \geq a_3 \geq \cdots$ *eventually stabilizes.*

Definition 2.50. *An* ***Artinian ring*** *is a ring that satisfies the descending chain condition on ideals. If a ring satisfies the descending chain condition on left ideals, then it is* **left Artinian. If a ring satisfies the descending chain condition on right ideals, then it is right Artinian.**

Example 2.51. *Any division ring must be right Artinian. This follows from the fact that it has no non-trivial right ideals, and therefore satisfies the descending chain property. Likewise, such rings are also left Artinian.*

3 Finite Rings

Finite rings are probably one of the prettiest types of rings in existence. Finiteness is a powerful trait, as it makes surjectivity equivalent to injectivity for multiplication operators. In this section, we shall introduce theorems concerning the injectivity and surjectivity these operators in finite rings, and eventually use these tools to prove that all finite rings are classical.

Theorem 3.1. *Let* R *be a ring with unity.* $\mathcal{L}_a$ *is surjective if and only if* a *has a right inverse (i.e.* a *is a left unit). Likewise,* $\mathcal{R}_a$ *is surjective if and only if* a *has a left inverse (i.e.* a *is a right unit). Both* $\mathcal{L}_a$ *and* $\mathcal{R}_a$ *are surjective if and only if* a *is a unit.*

Proof: Suppose $\mathcal{L}_a$ is surjective. This implies that $\mathcal{L}_a(x) = ax = 1$ for some $x \in R$.

Thus, we can see that a must have a right inverse. Conversely if a has a right inverse, say $ab = 1$ ($b \in R$). Then for all $y \in R$, $\mathcal{L}_a(by) = a(by) = (ab)y = 1y = y$. Thus $\mathcal{L}_a$ is surjective. A similar proof works for right multiplication maps.

If a has both a left and right inverse, say $ab = 1$ and $ca = 1$ for some $b, c \in R$, then $b = (ca)b = c(ab) = c$ so a has a two sided inverse $b = c = a^{-1}$ and is a unit. ■

Having proven in Theorem 3.1 that surjectivity of left or right multiplication is linked with the units of the ring, we now turn our attention to an interesting link between injectivity and surjectivity.

Theorem 3.2. *In a ring R with unity, if left multiplication by a, $\mathcal{L}_a$, is surjective, then right multiplication by a, $\mathcal{R}_a$, is injective. Also, if $\mathcal{R}_a$ is surjective, then $\mathcal{L}_a$ is injective.*

Proof: Suppose $\mathcal{L}_a$ is surjective and let $\mathcal{R}_a(x) = 0$ from some $x \in R$. By Theorem 3.1, since $\mathcal{L}_a$ is a surjective, there is some $b \in R$ such that $ab = 1$. Then $\mathcal{R}_a(x) = xa = 0$, and $x = x1 = x(ab) = (xa)b = 0b = 0$. We can then conclude that $\ker(\mathcal{R}_a)$ is trivial. Now by a previous theorem, $\mathcal{R}_a$ must be injective.

By switching left and right multiplication in this proof, we can show that if $\mathcal{R}_a$ is surjective, then $\mathcal{L}_a$ is injective. ■

Now, we shall recall our previous conclusions and add in a bit of group theory relating to permutations in order to make a conclusion about the existence of a "sided" identity in a finite ring.

Theorem 3.3. *If a finite ring R has a non-zero non-left zero divisor, the R has a left identity. Likewise, if a finite ring has a non-zero non-right zero divisor, then it has a* *right identity.*

Proof: Suppose $a \neq 0$, and that a is not a left zero divisor. Then the map $\mathcal{L}_a$ is injective. Since R is a finite ring, we have that $\mathcal{L}_a$ is onto. Thus, we can say that $\mathcal{L}_a$ is a permutation. Therefore, we find that there exists $N > 0$ such that $(\mathcal{L}_a)^N$ equals the identity permutation (in particular, if $|R| = n$, then the number of permutations on R is $n!$ and so $N = n!$ will work by Lagrange's theorem). Consider an arbitrary element $x \in R$, then

$$x = (\mathcal{L}_a)^N(x) = (\underbrace{\mathcal{L}_a \circ \mathcal{L}_a \circ \cdots \circ \mathcal{L}_a}_{N-\text{times}})(x) = \underbrace{a(a \cdots (a}_{N-\text{times}}\,x) \cdots) = a^N x.$$

So $a^N x = x$, and this is true for all elements $x \in R$. Therefore, we must conclude that a^N is a left identity.

A similar proof works for the right handed case. ■

We shall now briefly summarize our results concerning left and right multiplication in a finite ring with unity. First, we have proven that $\mathcal{L}_a$ is injective if and only if a is not a left zero divisor. Likewise, $\mathcal{R}_a$ is injective, if and only if a is not a right zero divisor. These conclusions also hold in a finite ring without unity.

Working in any ring with unity, we also proved that $\mathcal{L}_a$ is surjective if and only if a has a right inverse (or equivalently a is a left unit), and $\mathcal{R}_a$ is surjective if and only if a has a left inverse (or equivalently a is a right unit). Combining these two results, we determined that both $\mathcal{L}_a$ and $\mathcal{R}_a$ are surjective if and only if a is a unit, since right and left inverses will match via associativity.

Our next conclusion then stated that when working in a ring with unity, $\mathcal{L}_a$ is surjective implies that $\mathcal{R}_a$ is injective, and similarly, $\mathcal{R}_a$ is surjective implies that $\mathcal{L}_a$ is injective.

Using this information, we can conclude that if a ring R has at least 1 nonzero non-left zero divisor, then we can conclude that some element behaves as a left unity. Therefore, we find that R has a left identity, 1_L. Note that if R has a left but not a right identity, R may have more than one left identity. The same is true for the right. This is just like a unit which is a left but not right unit can have more than one right inverse. However, if R has both a left and right inverse, they must coincide and this two-sided identity is unique.

In Remark 2.9, we made reference to the fact that a ring could have more than one sided inverse. The following is an example of such a case.

Example 3.4. *An example of a ring with more than 1 left identity can be seen using elements from the set $S = \{z, a, b, c\}$, we can define addition and multiplication as given in the following tables as given in Deskins'* Abstract Algebra *in Example 4 in section 7.1 and Example 1 in 7.2 [D]:*

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

*	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Notice that S does not have a right identity. If it did, it would has a single two-sided identity. Also, notice that we have 0, two left units (i.e. a, c), and a left zero divisor (i.e. b). On the other hand, all the non-zero elements are right zero divisors (otherwise, we would have a right identity).

If we transpose the multiplication table, we would have an example of a ring with more than 1 right identity.

We shall now prove a major theorem concerning finite rings.

Theorem 3.5. *Let R be a finite ring and $a \in R$. Then a is either zero, a left zero divisor, or a left unit (i.e. has a right inverse). Also, a is either zero, a right zero divisor, or a right unit. If R is a ring with unity, then a is either zero, both a left and right zero divisor, or a unit.*

Proof: Suppose $a \neq 0$. If a is not a left zero divisor, then R has a left identity, say 1_L. In this case, $\mathcal{L}_a$ is injective. Thus, because R is finite, $\mathcal{L}_a$ is surjective. This means a is a left unit. Similarly, if a is nonzero and not a right zero divisor, R must have a right identity and a must be a right unit.

Suppose R has a unity. Let $a \neq 0$. Either $\mathcal{L}_a$ is injective or not. If $\mathcal{L}_a$ is injective, its surjective. However, since R has unity, we also get that $\mathcal{R}_a$ is injective ($\mathcal{L}_a$ surjective implies $\mathcal{R}_a$ injective). But R is finite, so $\mathcal{R}_a$ is also surjective. Therefore, a is a (two-sided) unit.

Alternatively, $\mathcal{L}_a$ fails to be injective (a is a left zero divisor). But then $\mathcal{R}_a$ cannot be surjective (otherwise this would imply that $\mathcal{L}_a$ is injective). Therefore, $\mathcal{R}_a$ cannot be injective (a is a right zero divisor). Therefore, a is both a left and right zero divisor. ■

This theorem then states that every finite ring is both left and right classical. In other words, every finite ring is classical.

4 Formal Fractions

An important addition to the ranks of classical rings is the total ring of quotients. In fact, a commutative ring can always be embedded into a classical ring, though this is not necessarily true for noncommutative rings. The origin of the ring of quotients for a commutative ring goes back to Grell, a student of Emmy Noether, who in 1926 constructed the ring of quotients for an integral domain [CM].

The question of whether a noncommutative version exists would not be posed until van der Waerden's *Moderne Algebra* in 1931 [W]. The immediate answer was no. Using a different approach, O. Ore was able to modify the original construction and bypass the initial roadblocks and succeed in construction a ring of fractions for certain noncommutative rings [O].

4.1 Fractions in Commutative Rings

Let R be a commutative ring (possibly without 1) and S be a multiplicative subset of R (i.e. $a, b \in S$ implies that $ab \in S$) and assume S does not contain zero or any zero divisors. Let $RS^{-1} = \{r/s \mid r \in R, s \in S\}$ be the set of equivalence classes r/s determined by the equivalence relation: $(r_1, s_1) \sim (r_2, s_2)$ iff $r_1 s_2 = r_2 s_1$ defined on $R \times S$.

Lemma 4.1. *The relation $\sim$ is an equivalence relation. $\frac{r}{s}$ denotes the equivalence class* *of* (r, s).

Proof: We need to show that the relation is reflexive, symmetric, and transitive.

Reflexive: $(r_1, s_1) \sim (r_1, s_1)$ because $r_1 s_1 = r_1 s_1$.

Symmetric: If $(r_1, s_1) \sim (r_2, s_2)$, then $r_1 s_2 = s_1 r_2$. Flipping around the equality, $r_2 s_1 = r_1 s_2$, and therefore, $(r_2, s_2) \sim (r_1, s_1)$.

Transitive: Assume $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$. We have that $r_1 s_2 = r_2 s_1$ and $r_2 s_3 = r_3 s_2$. We shall now take the first of these equations and multiply through by s_3. We get $r_1 s_2 s_3 = r_2 s_1 s_3$. We shall now multiply the second equation by s_1. This yields $r_2 s_3 s_1 = r_3 s_2 s_1$. Now we shall set each equation equal to 0, and then add the two. This gives $r_1 s_2 s_3 - r_2 s_1 s_3 + r_2 s_3 s_1 - r_3 s_2 s_1 = 0$. The middle terms cancel, and leave us with $r_1 s_2 s_3 - r_3 s_2 s_1 = 0$. Now s_2 is not a zero divisor nor zero, allowing us to cancel it out. Thus we have $r_1 s_3 = r_3 s_1$, and the action is transitive. ■

We shall now define for RS^{-1} operations of addition and multiplication. Addition will be defined as $\frac{r}{s} + \frac{x}{y} = \frac{ry + sx}{sy}$, and multiplication will be defined as $\frac{r}{s} \cdot \frac{x}{y} = \frac{rx}{sy}$. Note that $sy \in S$ because $s, y \in S$ and S is multiplicative.

Lemma 4.2. *Addition and multiplication are well defined.*

Proof: Suppose $(r_1, s_1) \sim (r_2, s_2)$ and $(x_1, y_1) \sim (x_2, y_2)$, so $r_1 s_2 = s_1 r_2$ and $x_1 y_2 = y_1 x_2$. We want to show that $(x_1 s_1 + r_1 y_1)(y_2 s_2) = (x_2 s_2 + r_2 y_2)(y_1 s_1)$. To do this, we shall begin by expanding the left and right sides of the equation to $x_1 s_1 y_2 s_2 + r_1 y_1 y_2 s_2 =$

$x_2s_2y_1s_1 + r_2y_2y_1s_1$. Now, replacing x_1y_2 with x_2y_1 and r_1s_2 with r_2s_1 on the left side due to our initial equalities. Thus, we have $x_2y_1s_1s_2 + r_2s_1y_1y_2 = x_2s_2y_1s_1 + r_2y_2y_1s_1$. Shuffling elements produces equality, and we have that addition is well-defined.

Consider $(r_1x_1)(s_2y_2) - (r_2x_2)(s_1y_1) = (r_1x_1)(s_2y_2) - (r_2x_2)(s_1y_1) + [(r_2s_1)(x_1y_2) - (r_2s_1)(x_1y_2)] = (r_1s_2 - r_2s_1)(x_1y_2) + (x_1y_2 - x_2y_1)(r_2s_1) = 0 + 0 = 0$. Thus $(r_1x_1, s_1y_1) \sim (r_2x_2, s_2y_2)$. Therefore, multiplication is well defined. ■

Theorem 4.3. *RS^{-1} is a commutative ring with unity. Fix a particular, $b \in S$. Then the equivalence class d/d is the identity and the mapping $r \mapsto rd/d$ is an injective homomorphism. This allows us to identify R as a subring of RS^{-1}. Also, a/b is a unit when $a \in S$ and its inverse is b/a.*

Proof: Let $a/b, c/d, e/f \in RS^{-1}$ (where $a, c, e \in R$ and $b, d, f \in S$).

Commutative: Note that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \frac{cb+da}{db} = \frac{c}{d} + \frac{a}{b}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b}$ so that addition and multiplication are both commutative. Essentially commutativity of both these operations is inherited from the commutativity of R.

Addition is Associative: Addition is associative because

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{ad+bc}{bd} + \frac{e}{f} = \frac{(ad+bc)f + bde}{bdf}$$
$$= \frac{adf + b(cf+de)}{bdf} = \frac{a}{b} + \frac{cf+de}{df} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$$

Additive Identity: We have $\frac{a}{b} + \frac{0}{d} = \frac{a \cdot d + b \cdot 0}{b \cdot d} = \frac{ad}{bd}$. Notice that $a \cdot bd = b \cdot ad$ so $\frac{a}{b} = \frac{ad}{bd}$. Thus $0/d$ is an additive identity. Note that for any $d, f \in S$ we have

$0 \cdot f = 0 = 0 \cdot d$ so that $0/d = 0/f$ so 0 over anything in S yields the additive identity.

Additive Inverses: $\dfrac{a}{b} + \dfrac{-a}{b} = \dfrac{ab + b(-a)}{bb} = \dfrac{0}{bb} = \dfrac{0}{b^2}$, and thus we have additive inverses. Since addition is commutative, we have an abelian group under addition.

Multiplication is Associative: $\left(\dfrac{a}{b} \cdot \dfrac{c}{d}\right) \cdot \dfrac{e}{f} = \dfrac{ace}{bdf} = \dfrac{a}{b} \cdot \left(\dfrac{c}{d} \cdot \dfrac{e}{f}\right)$

Multiplicative Identity: $\dfrac{d}{d} \cdot \dfrac{a}{b} = \dfrac{da}{db} = \dfrac{a}{b}$.

Distributivity: We have already shown that $ad/bd = a/b$ for any $d \in S$. Notice that also $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{ab + bc}{b^2} = \dfrac{(a+c)b}{b^2} = \dfrac{a+c}{b}$. Therefore, because of commutativity,

$$\frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{c}{d} + \frac{e}{f}\right)\frac{a}{b} = \frac{acf + ade}{bdf} = \frac{acf}{bdf} + \frac{ade}{bdf} = \frac{ac}{bd} + \frac{ae}{bf}.$$

Therefore, we have a commutative ring with unity where $0 = 0/d$ and $1 = d/d$ for any $d \in S$.

Pick some $d \in S$ and define $\phi : R \to RS^{-1}$ by $\phi(r) = \dfrac{rd}{d}$. $\phi(r+s) = \dfrac{(r+s)d}{d} = \dfrac{rd + sd}{d} = \dfrac{rd}{d} + \dfrac{sd}{d} = \phi(r) + \phi(s)$. Also, $\phi(rs) = \dfrac{rsd}{d} = \dfrac{rsd^2}{d^2} = \dfrac{rd}{d} \cdot \dfrac{sd}{d} = \phi(r)\phi(s)$.

Suppose that $\phi(r) = 0$ then $\dfrac{dr}{d} = \dfrac{0}{d}$ because $0 = 0/d$ for any $d \in S$. Thus $drd = 0d$ so that $rd^2 = 0$. But d is not a zero divisor so it can be canceled off and so $r = 0$. Thus ϕ is an injective homomorphism.

Note that if R already has unity, 1, then $\phi(1) = \dfrac{1d}{d} = \dfrac{d}{d} = 1$ (in RS^{-1}).

Thus, R is can be identified with $\phi(R)$ and can be viewed as a subring contained in RS^{-1}.

Finally, we turn to inverses. Suppose that $a, b \in S$. Then $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$. Thus $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$. ■

Remark 4.4. *If S is exactly the set of all nonzero, nonzero divisors, then a/b is a unit exactly when $a \in S$. Why? If a/b is a unit, then there exists some $c \in R$ and $d \in S$ such that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = 1 = \frac{d}{d}$. Thus $acd = bd^2$ and so $ac = bd$ (because d is not a zero divisor). Since bd is a nonzero, nonzero divisor, the same must be true of ac and so that must be true of both a and c. This means $a \in S$.*

Theorem 4.5. *Let R be a commutative ring and let S be the set of nonzero, nonzero divisors. We call $T(R) = RS^{-1}$ the* **total ring of quotients**. *The total ring of quotients is classical. This gives us the characterization (identifying R with its subring in RS^{-1}) that (when R is commutative) R is classical if and only if $T(R) = R$.*

Proof: Fix some nonzero, nonzero divisor $d \in R$ as in the injective homomorphism defined in Theorem 4.3.

We already have shown that $T(R)$ is a commutative ring and that (given the above remark) a/b is a unit if and only if both a and b belong to S. If $a \notin S$, then a is either 0 so that $a/b = 0/b = 0$ or a is a zero divisor. If a is a zero divisor, there exists some $c \in R$ such that $c \neq 0$ and $ac = 0$. But then $\frac{a}{b} \cdot \frac{c}{b} = \frac{ac}{b^2} = \frac{0}{b^2} = 0$. Note that both a/b and c/b are nonzero. In particular, if $a/b = 0/b$ then $ab = 0b$ and so $a = 0$ (since b is not a zero divisor and thus can be canceled). Thus $T(R)$ is classical.

Suppose R is classical and let $a/b \in T(R)$. We have that b is a nonzero, nonzero divisor in R thus b is a unit in R. Notice then that $\frac{a}{b} = \frac{ab^{-1}}{bb^{-1}} = \frac{ab^{-1}d}{1d} = \phi(ab^{-1})$. So

$a/b \in \phi(R)$ $(= R)$.

Conversely, suppose $T(R) = R$. Every nonzero, nonzero divisor in R, say b, can be a denominator and so $\dfrac{d}{bd} \in T(R) = R$. Notice that $\dfrac{bd}{d} \cdot \dfrac{d}{bd} = \dfrac{bd^2}{bd^2} = 1$. So $b^{-1} = \dfrac{d}{bd}$ exists (i.e. b is a unit). Thus R is classical. ■

When we are dealing with an integral domain, then the total ring of quotients is a field, and is called the **field of fractions**, or **field of quotients**. For example, the field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$. Also, the field of fractions of $\mathbb{F}[x]$ (polynomials with coefficients in a field $\mathbb{F}$) is $\mathbb{F}(x)$ (rational polynomials with coefficients in $\mathbb{F}$).

4.2 Fractions in Noncommutative Rings

In order to recreate Ore's work in constructing the noncommutative ring of fractions, we must first recall that for us to have existence of a field of fractions, we must have an integral domain. We could then derive that for a field of fractions to exist, elements must be of the form rs^{-1}, where $r, s \in R$ and $s \neq 0$. We can then ascertain that if we were to have a field of fractions, we should be able to write any element $b^{-1}a$ as $(a')(b')^{-1}$. We need $b^{-1}a = (a')(b')^{-1}$ and so we must have $ab' = ba'$. Notice here that a or $b = 0$ does not create an issue. Thus for any $a, b \in R - \{0\}$, we need the existence of $a', b' \in R$ such that $ab' = ba'$ $(\neq 0)$. This leads us to an important condition for our ring.

Definition 4.6. *Let R be a domain (a ring with unity with no zero divisors possibly noncommutative). For all $a, b \in R - \{0\}$, $aR \cap bR \neq 0$. We call this condition on R the **right Ore condition**. Similarly $Ra \cap Rb \neq 0$ yields the* **left Ore condition**.

Theorem 4.7. *Let R be a domain with the right Ore condition. Then R can be embedded in a skew field K.*

Proof: [Sketch] Suppose that R is a right Ore domain, and let

$$S = R \times (R - \{0\}) = \{(a,b) \mid a,b \in R, b \neq 0\}.$$

Let us define an equivalence relation $\sim$ on S where

$$(a,b) \sim (a',b') \text{ if and only if for some } u,u' \in R, au = a'u' \text{ and } bu = b'u' \neq 0.$$

The idea to capture the property of fractions that if $a/b = a'/b'$ we should be able to find some common denominator to be able to see equality $a/b = (au)/(bu) = (a'u')/(b'u') = a'/b'$. Note that this is an equivalence relation because it is reflexive, symmetric, and transitive.

Reflexive: Take $u = u' = 1$ and get $a1 = a1$ and $b1 = b1$ so $(a,b) \sim (a,b)$.

Symmetric: Symmetry is shown from the fact that if $(a,b) \sim (a',b')$, then $au = a'u'$ and $bu = b'u' \neq 0$. Thus, we can flip the equalities around and have symmetry.

Transitive: Now, for transitivity, suppose $(a,b) \sim (a',b')$ and $(a',b') \sim (a'',b'')$. So $au = a'u'$, $bu = b'u' \neq 0$, and $a'v = a''v'$ and $b'v = b''v' \neq 0$. We know by our condition that we can find $s,s' \in R$ such that $s' \neq 0$ and $b'u's = b'vs'$. Furthermore, by the fact that R is an integral domain, we know that $b'vs' \neq 0$. We then conclude that $u's = vs'$. We are then left with $aus = a'u's = a'vs' = a''v's'$ and $bus = bu's =$

$b'vs' = b''v's' \neq 0$. Therefore, $(a, b) \sim (a'', b'')$.

Therefore, we have an equivalence relation. Denote the equivalence class of (a, b) by $\frac{a}{b}$.

Now let us define operations on this structure. First, we define addition by

$$\frac{a}{b} + \frac{a'}{b'} = \frac{ac + a'c'}{bc}$$

where $c \in R$ and $c' \in R - \{0\}$ such that $bc = b'c'$. Note that we are able to do this via the right Ore condition. Next, we define multiplication as

$$\frac{a}{b} \cdot \frac{a'}{b'} = \frac{ad'}{b'd}$$

where $d \in R$ and $d' \in R - \{0\}$ such that $a'd = bd'$.

As before we must show that the operations are well-defined. Then we must show that the ring axioms all hold. See [O] for details.

We shall call this ring K. K turns out to be a skew field, as we can see that $\frac{a}{b} \neq 0$ if and only if $a \neq 0$, and then $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.

The mapping $a \mapsto a/1$ defines an embedding of R into K.

Finally, if we let $\varphi : R \to L$ be any embedding of R into a skew field L, then $\{\varphi(a)\varphi(b)^{-1} \mid a \in R, b \in R - \{0\}\}$ forms a subfield of L isomorphic to K. This shows that K is the unique smallest skew field containing a copy of R. ∎

Thus we have constructed the ring of quotients for a noncommutative ring. Like the commutative version of the ring of quotients, we can also conclude that the noncommutative ring of quotients is classical from a simple deduction (every nonzero element in our ring of quotients is a unit).

As a note, a more general case than the one used above could be constructed. In our construction, recall that we began assuming that R was a domain. A similar construction can be used where R is not a domain bust is just a noncommutative ring. In such a case, S must be constructed so it does not contain any zero divisors. The Ore condition is still required, as it was for our proof, but another condition, known as reversibility, is also needed. Reversibility refers to the fact that for $a \in R$, if $s'a = 0$ for some $s' \in S$, then $as = 0$ for some $s \in S$. Combining reversibility and the Ore condition in a ring allow us to always construct a ring of quotients RS^{-1}. More details on this construction can be found in examples 10.3 – 10.6 Lam's *Lectures on Modules and Rings* on page 300 [L].

5 Noncommutative Classical Rings

While the construction of the noncommutative ring of fractions may seem complex and not a little confusing, it is not the strangest thing we can find when working with noncommutative rings. Very odd cases can appear in which elements behave as zero divisors on one side, and as units or identity elements on the other. Such behavior can be confusing, but also quite interesting.

Consider V, a vector space of countably infinite dimension with basis $\beta = \{v_1, v_2, \cdots\}$. Define R to be the endomorphisms (i.e. linear operators) on V.

In particular, $R = \text{End}(V)$ where $\varphi \in R$ is a linear transformation from V to itself. Since the sum of linear transformations is a linear transformation, composing two linear transformations yields a linear transformation, and the identity map is a linear transformation, we get that R is a ring with unity.

Define $a : V \to V$ by $a(v_i) = v_{i+1}$ $(i = 1, 2, ...)$ extending linearly to all of V. Similarly, define $b : V \to V$ by $b(v_i) = v_{i-1}$ (where $i = 2, 3, ...$) and $b(v_1) = 0$. Finally, define $c : V \to V$ by $c(v_1) = v_1$ and $c(v_i) = \mathbf{0}$ for all $i \geq 2$. Notice that a, b, and c are all non-zero elements of R.

Using the above definitions, it is easy to see that $(b \circ a)(v_i) = b(v_{i+1}) = v_{i+1-1} = v_i$. Therefore, we see that $b \circ a$ is the identity transformation (it's the identity on the basis β so it's the identity on all of V by linearity). Because this is true, we can then make two conclusions. First, a is injective. This is true because $a(\mathbf{y}) = a(\mathbf{z})$ implies that $(b \circ a)(\mathbf{y}) = (b \circ a)(\mathbf{z})$. Therefore, $\mathbf{y} = \mathbf{z}$. Second, b must be surjective, as if $\mathbf{y} \in V$, then $b(a(\mathbf{y})) = (b \circ a)(\mathbf{y}) = \mathbf{y}$.

Now, note that $c \circ a = 0_V$ (the zero transformation). This derives from the fact that for $i \geq 1$, $c \circ a(v_i) = c(v_{i+1}) = \mathbf{0}$ because $i + 1 \geq 2$.

We have just shown that a is a right zero divisor and a right unit (b is a left inverse for a).

Consider $d : V \to V$ defined by $d(v_i) = v_{i-1}$ for $i \geq 2$ and $d(v_1) = \mathbf{w}$ (for any fixed $\mathbf{0} \neq \mathbf{w} \in V$). Then we will still have that $d \circ a$ is the identity so that d is a left inverse of a. However, if we choose $\mathbf{w} \neq v_1$, $b \neq d$. So a has infinitely many left inverses!

We could also find elements in this ring which are left zero divisors and left units with infinitely many right inverses.

It turns out that any element of R which is injective will be a right unit and any element which is surjective will be a left unit. If a nonzero element fails to be injective, it will be a left zero divisor and if it fails to be surjective, it will be a right zero divisor. So the bijections in R are the units. Every other element is a zero divisor. Moreover, any one sided unit will have infinitely many one sided inverses and is also a zero divisor. A proof of this is a fairly straight forward generalization of the examples built above.

In summary, R is classical.

Example 5.1. *Let's give a concrete version of the above example. Let $V = \mathbb{R}[x]$ be the (infinite dimensional) vector space of real polynomials and let $R = \mathrm{End}(\mathbb{R}[x])$. Then $D = \dfrac{d}{dx} \in R$. Also, let $I(f(x)) = \int_0^x f(t)\,dt$ and then $I \in R$. Let $E(f(x)) = f(0)$ and so $E \in R$.*

Then $DI(f(x)) = f(x)$ by the fundamental theorem of calculus and since the derivative of a constant is zero, $DE(f(x)) = D(f(0)) = 0$. So D is a left unit and a left zero divisor.

We now journey into the matrix rings as we continue our look at noncommutative rings. It is well-known that given a ring R, $R^{n\times n}$ ($n \times n$ square matrices) is a ring under matrix addition and matrix multiplication. Furthermore, if R is a ring with unity, then so is $R^{n\times n}$. Even if R is commutative, $R^{n\times n}$ is not when $n > 1$ (unless $R = \{0\}$).

In the case that R is a commutative with unity, we find that matrix properties, particularly the determinant of a matrix, behave as they do for real matrices as introduced in any standard linear algebra course. This fact will be valuable to us as we prepare to work with rings of matrices.

We begin by noting a couple of properties from introductory linear algebra that will be valuable for us soon.

For all of the following discussion let R be a commutative ring with unity.

Theorem 5.2. *Let A and B be two matrices in $R^{n\times n}$. Then the determinant of AB is equal to the product of the determinant of A and the determinant of B:*

$$\det(AB) = \det(A)\det(B).$$

Furthermore, the determinant of the identity matrix is $\det(I_n) = 1$.

We will not prove this here. However, most any introductory linear algebra text contains a proof of this fact (working over the real numbers) which can be adapted to a proof over a general commutative ring with unity.

To describe inverses we need the classical adjoint.

Definition 5.3. *Let R be a commutative ring with unity, and let A be an $n \times n$ matrix whose entries come from R. Then the **classical adjoint**, denoted C is the matrix whose (i, j)-entries are given by $c_{ij} = (-1)^{i+j}\det(A_{ij})$ where A_{ij} is the the matrix A with row i and column j struck out.*

Theorem 5.4. *Let $A \in R^{n\times n}$ and C be its classical adjoint. Then*

$$C^T A = AC^T = \det(A)I_n.$$

Proof: This essentially just follows from Cramer's rule. For an example of this, see McCoy's *Rings and Ideals* pages 157-158 [M]. ■

Example 5.5. *For a 2×2 matrix,* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$C = \begin{bmatrix} \det(A_{11}) & -\det(A_{12})) \\ -\det(A_{21}) & \det(A_{22}) \end{bmatrix} = \begin{bmatrix} \det(d) & -\det(c) \\ -\det(b) & \det(a) \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Note that this gives us $C^T A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \det(A)I_2.$

This now leads us to an interesting conclusion if we note that when $\det(A)^{-1}$ exists, we find that for a 2×2 matrix, $A^{-1} = (ad-bc)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ which is a our familiar 2×2 inverse formula from introductory linear algebra.

This then leads us to conclude the following.

Theorem 5.6. *Let $GL_n(R)$ denote the invertible $n \times n$ matrices with entries in R. In other words, $GL_n(R)$ is the set of units of $R^{n \times n}$. We have that*

$$(R^{n \times n})^\times = GL_n(R) = \{A \in R \mid \det(A) \in R^\times\}.$$

Proof: If $\det(A)$ is a unit of R, then A^{-1} exists (by our classical adjoint formula). In fact, $A^{-1} = [\det(A)]^{-1}C^T$. Conversely, if A^{-1} exists, then $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Thus we have that $\det(A)^{-1}$ exists, and $\det(A)^{-1} = \det(A^{-1})$. ■

We now state a powerful theorem of Neal McCoy which was presented in his 1948 book *Rings and Ideals* [M].

Theorem 5.7. *Let $R \neq 0$ be a commutative ring with unity, and $A \in R^{n \times n}$. Then the following are equivalent:*

- $\mathrm{rank}(A) = n$.
- *the rows of A are linearly independent.*
- *the columns of A are linearly independent.*
- $\det(A)$ *is a nonzero non-zero divisor.*

Proof: See McCoy pages 158-160 [M]. We note that this theorem is actually a specific case of McCoy's more general theorem. ■

An interesting consequence of this theorem follows from the last item.

Corollary 5.8. *For any $0 \neq A \in R^{n \times n}$ (where R is a nonzero commutative ring with unity). If the determinant of A is zero or a zero divisor, then A is both a left and a right zero divisor.*

Proof: Let A be a matrix whose determinant is either zero or a zero divisor. By the above theorem, A has linearly dependent rows and columns. This is the same as stating that there exists a nonzero $\mathbf{v} \in R^{n \times 1}$ such that $A\mathbf{v} = \mathbf{0}$ (dependent columns) and there exists a nonzero $\mathbf{w} \in R^{1 \times n}$ such that $\mathbf{w}A = \mathbf{0}$ (dependent rows).

Let B be the square matrix whose first column is $\mathbf{v}$ and all other columns are $\mathbf{0}$. Let C be a the square matrix whose first row is $\mathbf{w}$ and whose other columns are $\mathbf{0}$. Then

$AB = 0$ and $CA = 0$. Thus A is both a left and a right zero divisor. ■

A second corollary results from this theorem. It is corollary is a powerful tool for recognizing if a matrix ring is classical.

Corollary 5.9. *A commutative ring with unity R is classical if and only if $R^{n\times n}$ is classical (n some positive integer).*

Proof: Note that if $R = 0$, then $R^{n\times n} \cong 0$. So the theorem trivially holds.

Let $R \neq 0$ is a classical ring and $A \in R^{n\times n}$. If $A = 0$, then we are done. If $A \neq 0$, then we consider $\det(A)$. The $\det(A)$ must be either 0, a zero divisor, or a unit, since R is classical. If $\det(A)$ is a unit, then we find that A^{-1} exists, and A is a unit. On the other hand, if $\det(A)$ is either zero or a zero divisor, then from our corollary above, A is both a left and a right zero divisor. So $R^{n\times n}$ is a classical ring!

Suppose that $R^{n\times n}$ is classical. Let $a \in R$. Consider the matrix $A = aI_n$. Then $\det(A) = a^n$. We know that if A is a unit, then $\det(A) = a^n$ is a unit and thus a is a unit. Suppose that $a \neq 0$ and a^n is not a unit. Then A cannot be a unit (since its determinant isn't a unit) and also $A \neq 0$, so A must be (both a left and a right) a zero divisor. Suppose that $AB = 0$ for some $0 \neq B \in R^{n\times n}$. Then some entry of B is nonzero, but when scaled by a becomes zero. Therefore, a is a left zero divisor. Likewise, a is a right zero divisor using the fact that A is a right zero divisor. ■

6 Rings with Chain Conditions

We now move into a special collection of conditions on rings. Recall that we earlier defined the ascending chain condition (i.e. Noetherian) and descending chain conditions (i.e. Artinian) on rings. The chain conditions on rings turn out to be an effective way of finding whether or not a ring is classical. In fact, when looking for classical rings, chain conditions are a first step to recognizing a classical ring.

Remark 6.1. *For this section, we shall assume that our rings have unity unless otherwise noted.*

Theorem 6.2. *Every commutative Artinian ring is classical. Also, a left Artinian ring is right classical and a right Artinian ring is left classical.*

Proof: Suppose a ring R is a commutative Artinian ring, and let $0 \neq x \in R$. Then we have that an descending chain of ideals, $R = (1) = (x^0) \supseteq (x^1) \supseteq (x^2) \supseteq \cdots$ will stabilize at some point, say $(x^k) = (x^{k+1})$. Let's choose $k \geq 0$ such that k is minimal. Thus, we have that $x^k = yx^{k+1}$ for some $y \in R$. Suppose $k = 0$. Then, $1 = yx$. Therefore, x is a unit. Suppose $k > 0$. Then, $x^k - yx^{k+1} = 0$. Hence, $(x^{k-1} - yx^k)x = 0$. Observe that $x^{k-1} - yx^k \neq 0$, as k was minimal. Thus, x is a zero divisor. Therefore, we have that x in R is either a unit or a zero divisor, and R is classical.

Notice that this proof shows that a left Artinian ring is right classical if we replace the principal ideals with left principal ideals: $(b)_L = \{rb \mid r \in R\}$. A slightly adjusted proof gives that right Artinian implies left classical. ■

Example 6.3. *All finite rings are both Artinian and Noetherian since they only have finitely many ideals (of any kind). This allows us to then recover our earlier conclusions about finite rings.*

We shall start by proving an interesting theorem that not only shows when quotients of principal ideal domains are classical, but also familiarizes the reader with the use of the chain conditions in a proof.

Theorem 6.4. *Let R be a PID. Then for any nonzero ideal I, R/I is Artinian and so R/I is classical.*

Proof: Let R be a principal ideal domain and $I = (a)$ an ideal of R.

If $a = 0$, then we would find that $R/(0) \cong R$, and R may fail to be classical, as in the case when $R = \mathbb{Z}$. If a is a unit, then $R/I = R/(a) = R/R \cong \{0\}$ so R/I is classical. Let's assume $a \neq 0$ and a is not a unit.

Let $\mathcal{J}_1 \supsetneq \mathcal{J}_2 \supsetneq \cdots$ be a strictly descending chain of ideals in R/I. Now, by ideal correspondence (sometimes called the Fourth Isomorphism Theorem or Lattice Isomorphism Theorem), $\mathcal{J}_j = J_j/I$ for some $I \subseteq J_j \triangleleft R$. R is a PID so $J_j = (b_j)$ for some $b_j \in R$. Then we have $(a) \subseteq (b_j) \subsetneq \cdots \subsetneq (b_3) \subsetneq (b_2) \subsetneq (b_1)$. Now b_j divides a, so the prime (or irreducible) factors of b_j are among those of a, and we have finitely many choices. Now since $(b_{j+1}) \subsetneq (b_j)$, b_j is a proper divisor of b_{j+1}. So, if a has m irreducible factors, the chain can have at most m steps, and therefore the chain must terminate. Thus R/I is Artinian. ■

It is interesting to see that if we try to replicate the above proof for a unique factorization domain, it fails. Notice that the finiteness condition (the DCC) relies on controlling the length of a chain using the number of irreducible factors of our ideal's generator. If an ideal has more than one generator (i.e. it is not principal) the argument will not work.

Example 6.5. *$\mathbb{Z}$ is Noetherian, but it is not an Artinian ring. Therefore, we have that Noetherian does not imply classical.*

Despite this fact, there is still an important relationship between Artinian and Noetherian rings.

Theorem 6.6. *Artinian implies Noetherian.*

For a proof of this theorem, see Cohn pages 64-65 [C]. In this case, we find that Noetherian is simply not a strong enough condition on a ring in order to make it a classical ring. Even though the Noetherian condition does not imply classical, the condition is quite useful in studying rings. This is particularly because it allows one to easily simplify the ideal structure in a ring. A particularly useful theorem about Noetherian rings follows.

Theorem 6.7. *A Noetherian ring R satisfies the descending chain condition on prime ideals.*

The above theorem leads to a discussion of lengths of possible chains of prime ideals. In fact, such a length is called the Krull dimension.

Definition 6.8. *For a commutative ring R, the* ***Krull dimension*** *of R refers to the maximum possible length of a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of distinct prime ideals in*

R. If R has arbitrarily long chains of distinct prime ideals, then we say that the Krull dimension is infinite.

Remark 6.9. *Some authors, such as Dummit and Foote, refer to Krull dimension simply as dimension [DF].*

Example 6.10. *A field has Krull dimension 0. This is apparent from the fact that the only proper ideal in a field is the zero ideal. A principal ideal domain that is not a field has Krull dimension 1, since the maximal ideals are those generated by prime elements.*

Furthermore, we shall define the Jacobson radical of a ring.

Definition 6.11. *The* ***Jacobson radical*** *of R is the intersection of all maximal ideals of R. It is denoted $\mathrm{Jac}(R)$.*

The Jacobson radical will help us to prove a major theorem about Krull dimension, but we shall first provide a couple of theorems about the Jacobson radical in preparation. To present our theorem about the Jacobson radical we need to introduce one more thing.

Definition 6.12. *Let R be a commutative ring with unity and $I \triangleleft R$. The* **radical** *of I is*

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}$$

One can easily show that $\sqrt{I} \triangleleft R$.

Notice that $\sqrt{0} = \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}$ is the set of all nilpotent elements of R. We call this $\sqrt{0}$ the **nilradical** *of R.*

Theorem 6.13. *Let $\mathcal{J}$ be the Jacobson radical of a commutative ring R with unity.*

- *If I is a proper ideal of R, then so is $(I, \mathcal{J})$, the ideal generated by I and J.*
- *The Jacobson radical contains the nilradical of R: $\sqrt{0} \subseteq \mathrm{Jac}(R)$.*
- *An element x belongs to $\mathcal{J}$ if and only if $1 - rx$ is a unit for all $r \in R$.*
- *(Nakayama's Lemma) If I is a finitely generated ideal of R and $\mathcal{J}I = I$, then $I = 0$.*

Proof: See Theorem 3 on page 751 of Dummit and Foote [DF]. ■

Theorem 6.14. *Let R be an Artinian ring. Then every prime ideal of R is maximal (i.e. R has Krull dimension 0).*

Proof: In order to prove this, we begin by proving that $\mathcal{J} = \mathrm{Jac}(R)$ is nilpotent. By the descending chain condition, we know that there must exist $m > 0$ such that $\mathcal{J}^m = \mathcal{J}^{m+i}$ for all $i > 0$. For sake of contradiction, suppose $\mathcal{J} \neq 0$, and let $\mathcal{P}$ be the set of all proper ideals I satisfying the condition that $I\mathcal{J} \neq 0$. This then gives us that the Jacobson radical must be an element of $\mathcal{P}$. Since we have the descending chain condition, we have a minimal element in $\mathcal{P}$, say I_0. Now there exists some $x \in I_0$ such that $x\mathcal{J} \neq 0$, and by minimality, we see that $I_0 = (x)$. However, we find that $((x)\mathcal{J})\mathcal{J}^m = x\mathcal{J}^{m+1} = x\mathcal{J}^m$. Thus we have that $(x) = (x)\mathcal{J}$, since (x) was minimal. Thus, using Nakayama's Lemma, $(x) = 0$, but that is a contradiction! Therefore, the Jacobson radical is nilpotent.

Since the Jacobson radical is nilpotent, we have that $\mathrm{Jac}(R) \subseteq \sqrt{0}$. Thus we have that the two ideals are equal by our previously stated theorem.

Now, every prime ideal P in R contains the nilradical of R, and thus contains $\mathrm{Jac}(R)$. So the image of P will be a prime ideal in the quotient ring $R/\mathrm{Jac}(R)$. Now this quotient ring is isomorphic to $\mathbb{K}_1 \times \cdots \times \mathbb{K}_n$ (each $\mathbb{K}_i$ is a field), which is a result from the Chinese Remainder Theorem applied to the maximal ideals in R (The intersection of these maximal ideals is the Jacobson radical). Note that in a direct product of rings, $R \times S$, we have that all ideals are of the form $I_R \times I_S$. It then follows that one of the prime ideals in $\mathbb{K}_1 \times \cdots \times \mathbb{K}_n$ will consist of elements that are 0 in one of the components. This ideal will also be a maximal ideal in $\mathbb{K}_1 \times \cdots \times \mathbb{K}_n$, and we then have that P must have been a maximal ideal in R. ■

Corollary 6.15. *A ring R is Artinian if and only if R is Noetherian and has Krull dimension 0.*

Proof: [Sketch] The first half of this proof was completed previously. For the converse, suppose that R is a Noetherian ring with Krull dimension 0. Then the prime ideals are maximal. Since R is Noetherian, we can then say that $(0) = P_1 \cdots P_n$ is the product of primes. These primes are not necessarily distinct. Since we have Krull dimension 0, these primes must be maximal. Now we can apply the Chinese Remainder Theorem, and we find that R is isomorphic to the direct product of a finite number of Noetherian rings of the form R/M^m, where M is a maximal ideal. It remains to show that M is maximal in R. For complete details see [DF] page 752–753 Corollary 4. ■

This argues that being Artinian is very close to having Krull dimension 0. We know that Artinian rings are classical. We could ask if having Krull dimension 0 is enough to guarantee a ring is classical. The answer is yes.

First we need two other theorems that will be necessary in order to show that a ring is classical. These theorems are due to Krull's work.

Theorem 6.16. *(Krull's Theorem) Let R be a Noetherian ring and r an element of R which is neither a zero divisor nor a unit. Then every minimal prime ideal P containing r has height 1.*

Proof: See Theorem 142 of Kaplansky's *Commutative Algebra* on pages 104-105 [K]. ■

Theorem 6.17. *Let R be a Noetherian ring, $I \triangleleft R$, M be a finitely generated R-module, and $B = \cap_{n=1}^{\infty} I^n M$. Then $IB = B$.*

Proof: See Theorem 74 of Kaplansky's *Commutative Algebra* on pages 48-49 [K]. ■

Theorem 6.18. *Let R be a commutative ring with unity which has Krull dimension 0. Then R is classical.*

Proof: [Sketch] If R has Krull dimension 0, then we know that any maximal ideal M of R is also a minimal prime ideal by definition. Applying Krull's theorem on the intersection of prime ideals to the localization R_M, we find that MR_M is the nilradical of R_M. So for any $f \in M$, there exists a positive integer n such that $f^n = 0$ in R_m. So there exists $s \in R/M$ such that $sf^n = 0$ in R. We can choose the smallest n with respect to this property so that $sf^{n-1} \neq 0$. Therefore f is a zero divisor. Now any non-unit element f belongs to some maximal ideal, it is a zero divisor. Therefore, we have that all elements are either zero, zero divisors, or units, and R is classical. ■

Remark 6.19. *For example,* $\mathbb{R}[x]$ *has Krull dimension 1 and is not a classical ring. However, it turns out that classical does not imply Krull dimension 0. In fact, there are classical rings without Krull dimension 0 (we will not construct any here). There are other more general conditions which imply that a ring is classical. Lam's* Lectures on Modules and Rings *[L] discusses several such conditions. In particular, page 321 discusses strongly* π*-regular rings which are exactly those of Krull dimension 0 in the commutative case, but not so for noncommutative rings.*

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