Poset Diagrams for Twisted Involutions of Weyl Groups

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Abstract

Representation theory of symmetric spaces is an increasingly important field of mathematics that gives useful insight into many areas of science and technology. To better understand generalized symmetric spaces for the special Coxeter groups known as Weyl groups, we investigate the symmetric group, the Weyl groups of the special linear Lie algebras. Given a Weyl group W and an involution θ, we define the extended symmetric space of W as

$$J_\theta = \{w \in W | \theta(w) = w^{-1}\}.$$

In studying this extended symmetric space, we have identified a specific process by which to construct visual representations of the space for any θ-twisted involution of S_4. We were able to show an isomorphic relationship between each diagram of S_4 generated in this way, which we then extended to give an isomorphic relationship between each diagram for θ-twisted involutions in S_n.

1 Background

The field of Lie Theory relies heavily on root-space decomposition of Lie algebras. A Lie algebra can be characterized by its root system, and the root systems of these algebras can be represented using Weyl groups. Each Weyl group can be generated by simple reflections, which are elements of order two. Every finite semi-simple Lie algebra can be constructed using only the special linear algebra $sl(n)$, so we are motivated to study the Weyl groups which represent $sl(n)$, known as the symmetric group S_n.

We begin our discussion by introducing a very special and useful group known as the symmetric group, S_n.

Definition 1.1. [2] *Given the set $A = \{1, 2, 3, ..., n\}$, a permutation on A is a bijection from A to itself. The set of all permutations of A, together with the operation of function composition, is called* **the symmetric group of degree n** *and is denoted S_n.*

Elements of S_n can be written in the form

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \ldots n \\ \sigma(1) & \sigma(2) & \sigma(3) \ldots \sigma(n) \end{pmatrix}.$$

However, there exists an even more convenient way of writing σ, known as cycle notation. Consider the following example of a permutation, α:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}.$$

Notice that α will send 1 to 2 and 2 to 1, thus the action of α on the elements 1 and 2 can be expressed as $(1, 2)$ or as $(2, 1)$. However, for the sake of simplicity, we will begin each cycle with the lowest possible number. If we represent α's action on the set of elements $\{1, 2, 3, 4, 5, 6\}$ similarly, the result is $\alpha = (12)(346)(5)$. Note: In this example, α sends 5 to itself, thus the element (5) is considered the identity element. In fact, every cycle of length 1 is the permutation identity, so we will supress the 1-cycles in our expressions, so that $\alpha = (12)(346)$.

Definition 1.2. *Given a permutation α of S_n, we say that the* **order** *of α equals n (denoted $|\alpha| = n$) if n is the smallest possible integer such that $\alpha^n = (1)$.*

Recall the element of S_4 given previously, (12). Since $(12)^2 = (12)(12) = (1)$, we say that $|(12)| = 2$. In fact, every single two-cycle permutation is of order 2, and every single n-cycle permutation is of order n. Another property of permutations is that every permutation can be written as a product of disjoint cycles. Since all disjoint cycles commute, the order of any product of disjoint cycles is the least common multiple of the orders of each individual cycle. For example, the permutation $\alpha = (12)(346)(5)$ is of order 6. We know this by observing that $|(12)| = 2, |(346)| = 3$, and $|(5)| = 1$. Since each cycle is disjoint, $|\alpha| = \text{lcm}\{1, 2, 3\} = 6$.

Example 1.3. *To demonstrate the composition of two permutations, consider the following. Let*

$$\alpha = (15)(243) \text{ and } \beta = (154)(23)$$

be permutations of the group S_5. The composition of α and β is defined as

$$\begin{aligned} \beta\alpha &= (154)(23)(15)(243) \\ &= (142), \end{aligned}$$

so $|\alpha| = |\beta| = 6$, while $|\beta\alpha| = 3$.

Definition 1.4. *An* **automorphism** *of a group G is an isomorphism from G to itself.*

The automorphisms of S_n are typically easy to deal with, the exception being a single outer automorphism in S_6. If we restrict our focus to $S_{n\neq 6}$, then every automorphism of the symmetric group is an inner automorphism.

Definition 1.5. *Let G be a group, and let $x \in G$ be given. The function ϕ_x defined by $\phi_x(g) = xgx^{-1}$ for any $g \in G$ is called the* **inner automorphism** *of G induced by x.*

Thus for the automorphisms of $S_{n\neq 6}$, the relationship between elements is conjugation. This motivates the next definition.

Definition 1.6. *Given a group G, the* **conjugacy class** *of each $g \in G$ is defined as*

$$K_g = \{\phi_x(g) | x \in G\}.$$

The structure of S_n is critical in representation of Lie algebras, and conjugacy classes allow the group to be partitioned in an easily predictable way. This is a result of the fact that conjugacy preserves cycle type.

Theorem 1.7. [1] *Two permutations in S_n are conjugate if and only if they the same cycle type decomposition.*

Proof. Consider the permutations g and x, where $|g| = n$. Now let y be the conjugacy of g by x, so that $y = xgx^{-1}$. Now

$$y = xgx^{-1} \Rightarrow y^n = (xgx^{-1})^n = xg^n x^{-1},$$

thus $|y| = |g|$ for all $g, x \in S_n$. □

Definition 1.8. *Any element of a group that has order 1 or 2 is an* **involution**. *Additionally, any automorphism of order 1 or 2 is said to be* **involutory**.

Recall that we have restricted our focus to $S_{n\neq 6}$, where each automorphism is defined by the familiar conjugation

$$\theta_z(g) = zgz^{-1}, \text{where } z, g \in S_{n\neq 6}.$$

Furthermore, the set of involutory θ_z's imposes a restriction on the choice of conjugating elements z. Note that for θ_z to be involutory, we must have that

$$\theta_z^2(g) = g \text{ for all } g \in S_{n\neq 6}.$$

To this end, consider the following proposition.

Proposition 1.9. *For each inner automorphism of $S_{n\neq 6}$, $\theta_z(g)$ is involutory if and only if z is an involution.*

Proof. First, suppose that $\theta_z(g)$ is involutory. Then

$$\begin{aligned} \theta_z^2(g) &= g \\ \Rightarrow z(zgz^{-1})z^{-1} &= g \\ \Rightarrow z^2gz^{-2} &= g \\ \Rightarrow z^2g &= gz^2 \ \forall\ g \in S_n. \end{aligned}$$

So we have that z^2 commutes with every $g \in S_n$, thus z^2 is in the center of S_n. Since the only element in the center of the symmetric group is the identity element, $z^2 = (1)$. Therefore z is an involution.

Next, suppose that z is an involution, so $z^2 = (1)$. Then

$$\begin{aligned} \theta_z^2(g) &= z(z(g)z^{-1})z^{-1} \\ &= z^2gz^{-2} \\ &= z^2g(z^2)^{-1} \\ &= (1)g(1)^{-1} \\ &= (1)g(1) \\ &= g. \end{aligned}$$

Thus θ_z is clearly involutory and our proof is complete. □

Definition 1.10. [3] *Given a group G and the fixed point subgroup H of G under θ, where H is defined as*

$$H = \{g \in G | \theta(g) = g\},$$

we will define the **generalized symmetric space** *of G as the set of all cosets G/H.*

As discussed in [4], involutory automorphisms lend a convenient way to generalize symmetric algbraic spaces. For some group G, consider the automorphism $\tau : G \to G$ defined by $\tau(g) = g\theta(g)^{-1}$ for some $g \in G$ and some involutory map θ, and define the image of τ on G as $Q = \{g\theta(g)^{-1} | g \in G\}$. Also consider the fixed point subgroup H of G, where $H = \{g \in G | \theta(g) = g\}$. Q is a closed subvariety of G and τ induces an isomorphism of the coset G/H onto Q, so $G/H \cong Q$.

Definition 1.11. *Let θ be an involution on the group G. The extended symmetric space of G is defined as*

$$J_\theta = \{g \in G | \theta(g) = g^{-1}\}$$

Note that, by definition, $G/H \subset J_\theta$, since $G/H \cong Q = J_\theta$. In studying the extended symmetric space, our results will carry over to the generalized symmetric space.

2 Constructing the Extended Symmetric Space of S_n

In 2012, Haas and Helminck introduced an algorithm for generating the extended symmetric space for S_n in the form of a diagram, giving an easily visible structure to the group. We begin by defining a particular form of conjugation that generates the extended symmetric space.

Definition 2.1. *For any involution θ on G, the action $*$ defined by*

$$x * g = xg\theta(x)^{-1}$$

is called θ **-twisted conjugation***.*

Example 2.2. *Consider the involution $\theta_{(13)}$ on S_4 defined by $\theta_{(13)}(g) = (13)g(13)^{-1}$. The*

θ-twisted conjugation on the element (123) *by the conjugating element* (12) *is*

$$\begin{aligned}(12) *_{(13)} (123) &= (12)(123)\theta_{(13)}(12)^{-1} \\ &= (12)(123)(13)(12)^{-1}(13)^{-1} \\ &= (12)(123)(23) \\ &= (1)\end{aligned}$$

As opposed to basic conjugation where cycle-type is preserved, the previous example clearly shows that θ-twisted conjugation does not preserve cycle-type. To visualize the structure imposed by this type of conjugation, Haas and Helminck describe an algorithm to generate a diagram representing these special conjugacy classes.

Definition 2.3. *Given a generating set $\langle s_1, s_2, ..., s_{n-1}\rangle$ of S_n, where every generating element is of order two, the* **poset diagram** *induced by θ-twisted conjugation for some $z \in S_n$ is given as follows:*

Begin with some initial vertex $v_i \in S_n$. Perform θ-twisted conjugation on v_i by every s_i in the generating set, and denote this action as $\bar{s}_i(v_j)$. If

$$\begin{aligned}\bar{s}_i(v_j) &= s_i *_z v_j \\ &= s_i v_j \theta_z(s_i)^{-1} \\ &= s_i v_j [z s_i z^{-1}]^{-1} \\ &= s_i v_j z s_i z \text{ (since } |s_i| = 2 \text{ and } |z| = 1 \text{ or } 2) \\ &= v_1\end{aligned}$$

If $\bar{s}_i(v_j) = v_1$ where $v_j \neq v_1$, then v_1 becomes a new vertex in the diagram, connected to v_j by a solid line representing the conjugating value s_i. However, if $\bar{s}_i(v_j) = v_j$, then we perform left-multiplication on v_j by s_i, so that

$$\begin{aligned}s_i(v_j) &= s_i v_j \\ &= v_1\end{aligned}$$

is a new vertex connected to v_j by a dashed line representing s_i. Once s_i is exhausted for all vallues of i, we repeat this process at each of the new vertices until there are no new vertices. We follow this description with an example that serves as the basic template for the remainder of our discussion on poset diagrams.

Example 2.4. *Consider the symmetric group S_4. Choose the generating set*

$$\langle s_1, s_2, s_3 \rangle = \langle (12), (23), (34) \rangle ,$$

and let $z = (1)$. Beginning at the vertex $v_j = (1)$, we will perform (1)-twisted conjugation on (1) with each of the generating elements. Since

$$\begin{aligned} \overline{s_1}((1)) = (12)(1)(1)(12)^{-1}(1) &= (1) \\ \overline{s_2}((1)) = (23)(1)(1)(23)^{-1}(1) &= (1) \\ \overline{s_3}((1)) = (34)(1)(1)(34)^{-1}(1) &= (1), \end{aligned}$$

we then multiply each generator with (1), resulting in the new vertices

$$\begin{aligned} s_1((1)) = (12)(1) &= (12) \\ s_2((1)) = (23)(1) &= (23) \\ s_3((1)) = (34)(1) &= (34). \end{aligned}$$

Then repeat the process on each of the new vertices:

$$\begin{aligned} \overline{s_1}((12)) = (12)(12)(1)(12)^{-1}(1) &= (12) \Rightarrow s_1((12)) = (1) \\ \overline{s_2}((12)) = (23)(12)(1)(23)^{-1}(1) &= (13) \\ \overline{s_3}((12)) = (34)(12)(1)(34)^{-1}(1) &= (12) \Rightarrow s_3((12)) = (12)(34) \end{aligned}$$

$$\overline{s_1}((23)) = (12)(23)(1)(12)^{-1}(1) = (13)$$
$$\overline{s_2}((23)) = (23)(23)(1)(23)^{-1}(1) = (23) \Rightarrow s_2((23)) = (1)$$
$$\overline{s_3}((23)) = (34)(23)(1)(34)^{-1}(1) = (24)$$

$$\overline{s_1}((34)) = (12)(34)(1)(12)^{-1}(1) = (34) \Rightarrow s_1(34) = (12)(34)$$
$$\overline{s_2}((34)) = (23)(34)(1)(23)^{-1}(1) = (24)$$
$$\overline{s_3}((34)) = (34)(34)(1)(34)^{-1}(1) = (1)$$

Continue repeating the algorithm until all new vertices have been exhausted and the following diagram will result.

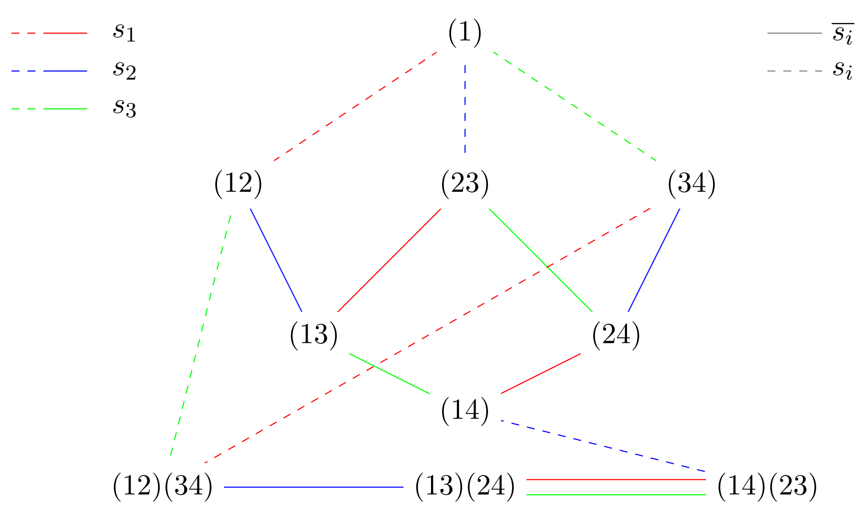

Figure 1: $S_4 = \langle (12), (23), (34) \rangle$

The astute reader will observe that this is indeed equivalent to basic conjugation, since our twisting value is (1). As a result, cycle-type is preserved throughout the algorithm, and thus the conjugacy classes of the extended symmetric space of S_4 are clearly revealed by the diagram. Notice the heirarchic prioritization, where all the one-cycles of S_4 are listed in the first row, the two-cycles are listed in the second, third, and fourth rows, and the double-two-cycles are listed in the last row. Furthermore, every element is connected via a solid line to another element of equal cycle-type. In order to move between conjugacy classes (that is, to move between two elements of different cycle-type), we must follow a dashed line representing left-multiplication as opposed to conjugation.

As is obvious from Example 4.2, θ-twisted conjugation need not preserve cycle type, thus the conjugacy classes are not intuitive and would be tedious to consider without the algorithm and the poset diagram.

Example 2.5. *Once again considering the symmetric group S_4, choose the generating set $\langle (12), (23), (34) \rangle$, and let $z = (12)$. Beginning at the vertex $v_j = (1)$, the diagram induced is:*

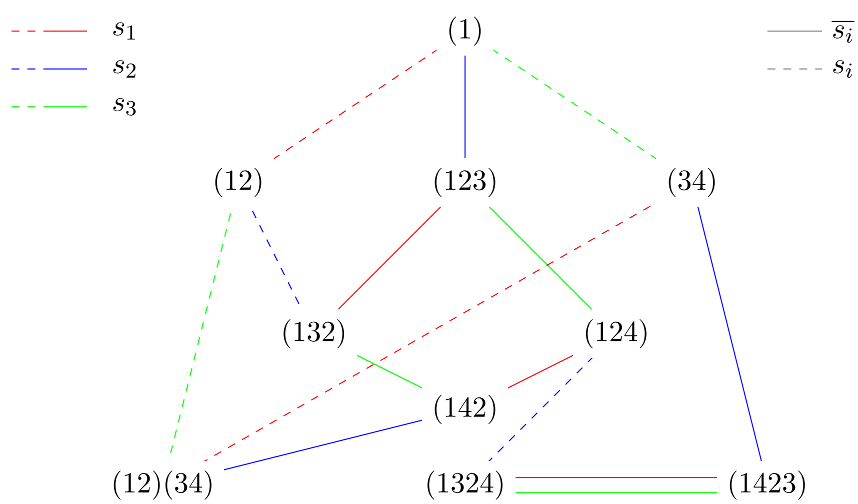

Figure 2: (1)-twisted Conjugation Starting at (1) with Partial Ordering

Once again, the conjugacy classes are clearly visible in the diagram. However, the partitioning would not be clear without the diagram. Here we have that one-,three-, and double-two-cycle permutations are all contained within the same conjugacy class, while two- and four-cycle permutations are in seperate classes.

In studying the diagrams induced by θ-twisted conjugation, the observation was made fairly quickly that this method is tedious and time-consuming. Might there be a way to partition these subgroups by simply observing the twisting value of θ? The main objective of our research is to answer this question. In order to do so we studied several different partially ordered diagrams in search of some pattern that might enlighten us, and to do so we unravelled the diagrams so that there would be no intersecting paths. While these diagrams were no longer partially ordered, the partitioning of conjugacy classes was perserved and more clearly visible to the eye. Consider the following rearrangement of vertices of the diagrams presented in **Example 3.4** and **Example 3.5**.

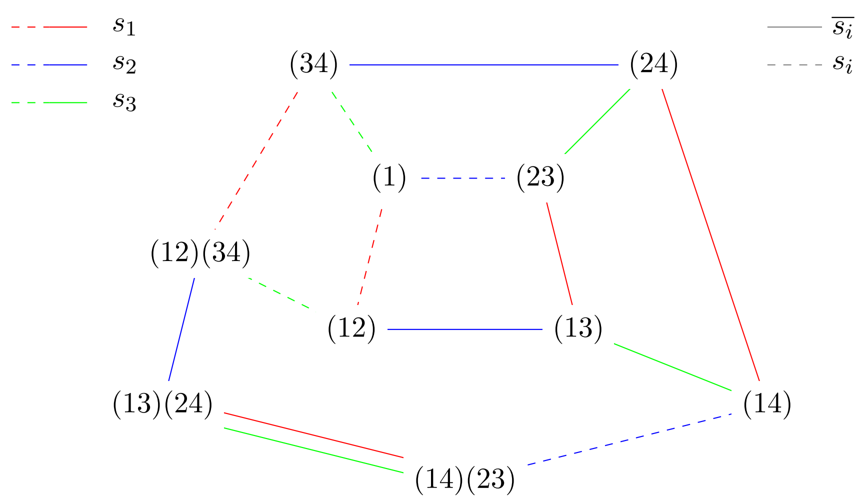

Figure 3: (1)-twisted Conjugation Starting at (1) without Partial Ordering

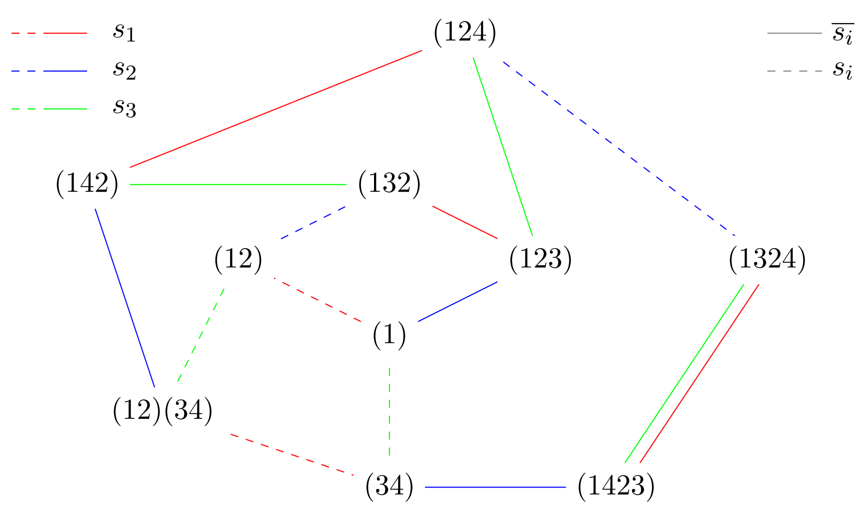

Figure 4: (12)-twisted Conjugation Starting at (1) without Partial Ordering

Our next task is to show that (12)-twisted conjugation is isomorphic to the basic (1)-twisted conjugation. First, note that all the vertices in the (12)-twisted diagram can be obtained by right-multiplying each vertex in the basic diagram by the twisting value $z = (12)$. Next, recon-

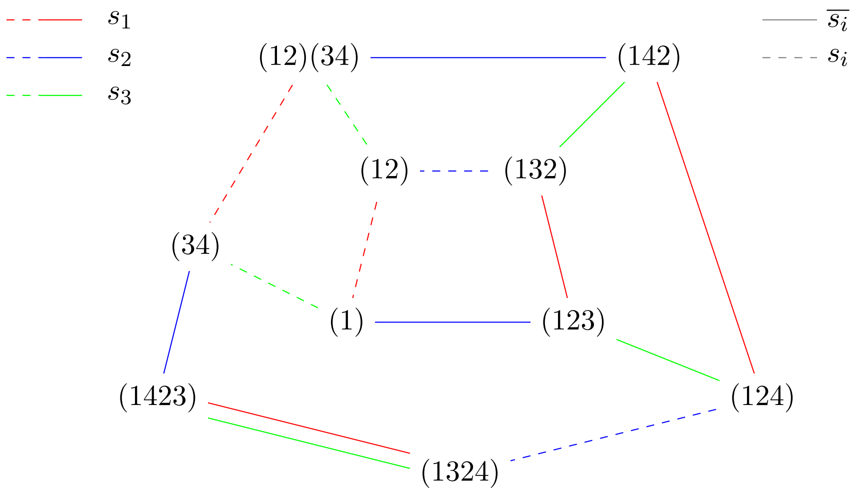

Figure 5: Example 3.4 with each vertex right-multiplied by (12)

sider the diagram induced by (12)-twisted conjugation starting at $v_1 = (1)$. Before performing the algorithm for poset diagrams, replace the initial vertex (1) with $(1)(12) = (12)$, then complete the diagram as we did previously. The result is certainly familiar.

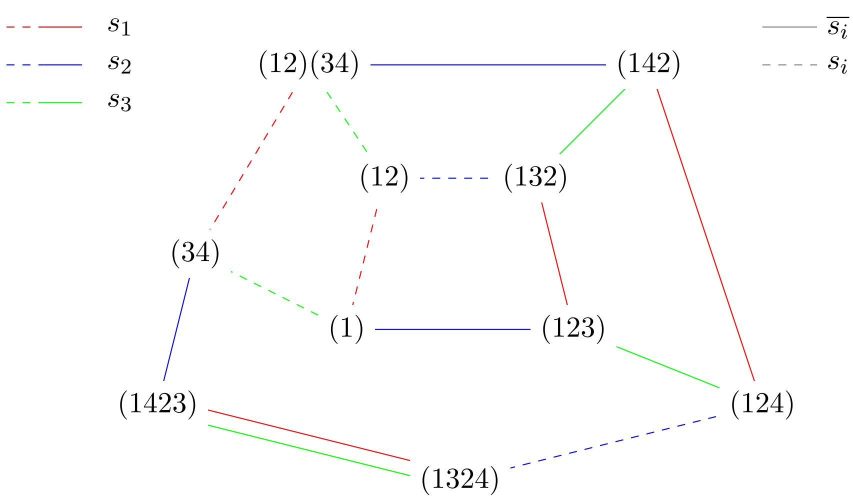

Figure 6: (12)-twisted Conjugation Starting at $(1)(12) = (12)$

This relationship begins our attempt to show a graph isomorphism between the diagrams induced by basic conjugation and any θ-twisted conjugation in $S_{n\neq 6}$.

3 Graph-Isomorphism

In order to avoid the tedious computation of generating numerous diagrams, and to guarantee that our results will apply to all S_n for any value $n \neq 6$, a graph-isomorphism is needed.

Proposition 3.1. *Given a symmetric group of inner automorphisms $S_{n\neq 6}$ and a generating set of order two elements $\langle s_1, s_2, ..., s_{n-1}\rangle$, every diagram induced by θ-twisted conjugation is isomorphic to the diagram induced by basic conjugation beginning at the identity.*

Proof. Let $\phi : \Gamma_{(1)} \to \Gamma_z$ be given by $\phi(v_i) = v_i z$, where Γ_* is the vertex set of the $*$-twisted poset diagram. Since ϕ is defined by right-multiplication (a group action), ϕ is clearly a bijection, simply rearranging the elements of the vertex set. Now suppose we have some $v_i, v_j \in \Gamma_{(1)}$ (where $v_i \neq v_j$) such that v_i is connected to v_j via the operation $\overline{s_t}$, so

$$\begin{aligned} \overline{s_t}(v_i) &= s_t v_i s_t^{-1} \\ &= v_j. \end{aligned}$$

We want to show that, since v_i connects to v_j in $\Gamma_{(1)}$ via $\overline{s_t}$, $\phi(v_i) = v_i z$ also connects to $\phi(v_j) = v_j z$ in Γ_z via $\overline{s_t}$. So

$$\begin{aligned} \overline{s_t}(\phi(v_i)) &= s_t(v_i z) z s_t^{-1} z^{-1} \\ &= s_t v_i s_t^{-1} z \text{ (since z is an involution)} \\ &= v_j z \\ &= \phi(v_j). \end{aligned}$$

So ϕ is an isomorphism from all the new vertices in $\Gamma_{(1)}$ obtained via $\overline{s_t}$ to all similarly obtained vertices in Γ_z. Next we will consider all the new vertices of $\Gamma_{(1)}$ obtained via s_t. Suppose that v_i connects v_j via s_t, that is,

$$\overline{s_t}(v_i) = v_i \Rightarrow s_t(v_i) = v_j.$$

Now

$$\begin{aligned}\overline{s_t}(\phi((v_i)) &= s_t(v_i z) z s_t^{-1} z^{-1} \\ &= v_i z \\ &= \phi(v_i),\end{aligned}$$

thus we are required to perform $s_t(\phi(v_i))$ to result in the connecting vertex. To this end,

$$\begin{aligned}s_t(\phi(v_i)) &= s_t v_i z \\ &= v_j z \\ &= \phi(v_j).\end{aligned}$$

So for every $v_i, v_j \in \Gamma_{(1)}$ connected via s_t, $\phi(v_i)$ is connected to $\phi(v_t)$ via the same s_t. Thus ϕ is an isomorphism from all vertices in $\Gamma_{(1)}$ to the vertices in Γ_z and the proof is complete. $\square$

Example 3.2. *Given the generating set $S_5 = \langle (12), (23), (34), (45) \rangle$, find the twisted conjugacy classes of S_5 for the twisting element $z = (15)$.*

Solution. Since basic conjugation preserves cycle type, we begin by listing the basic conjugacy classes of S_5, an easy task since we only need the order-2 elements of S_5:

$$\begin{aligned}K((1)) &= \{(1)\} \\ K((12)) &= \{(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\} \\ K((12)(34)) &= \{(12)(34), (12)(35), (12)(45), (13)(24), (13)(25), (13)(45), (14)(23), (14)(25), \\ &\quad (14)(35), (15)(23), (15)(24), (15)(34), (23)(45), (24)(35), (25)(34)\}\end{aligned}$$

By **Proposition 3.1**, the (15)-twisted conjugacy class of S_5 is simply

$$\begin{aligned} K((1)(15)) &= \{(1)(15)\} = \{(15)\} \\ K((12)(15)) &= \{(12)(15),(13)(15),(14)(15),(15)(15),(23)(15),(24)(15),(25)(15), \\ &\quad (34)(15),(35)(15),(45)(15)\} \\ &= \{(152),(153),(154),(1),(15)(23),(15)(24),(125),(15)(34),(135),(145)\} \end{aligned}$$

$$\begin{aligned} K((12)(34)(15)) &= \{(12)(34)(15),(12)(35)(15),(12)(45)(15),(13)(24)(15),(13)(25)(15), \\ &\quad (13)(45)(15),(14)(23)(15),(14)(25)(15),(14)(35)(15),(15)(23)(15), \\ &\quad (15)(24)(15),(15)(34)(15),(23)(45)(15),(24)(35)(15),(25)(34)(15)\} \\ &= \{(152)(34),(1352),(1452),(153)(24),(1253),(1453),(154)(23),(1254), \\ &\quad (1354),(23),(24),(34),(145)(23),(135)(24),(125)(34)\} \end{aligned}$$

◀

4 Conclusion

The algorithm presented by Haas and Helminck proves to be a useful tool for revealing the partitioning of a permutation group via conjugacy classes. However the diagrams are tedious and time-consuming to create, so in an attempt to shorten the process of generating conjugacy classes we have observed a graph isomorphism between diagrams generated by basic conjugation and those generated by twisted conjugation. Since our observations hold for the extended symmetric spaces of groups, they will also hold for the generalized symmetric spaces. This guarantees that our work in S_n will also apply to more naturally occuring symmetries in S_n, such as the Weyl groups. With this new tool at our disposal we hope to continue our research by observing the structure of special subgroups of S_n known as Weyl groups. In closing I would like to give special thanks to Lee Fisher, Aneisy Cardo, Nadine Lambert, and Dr. Vicki Klima for all their help and guidance in our research.

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