Positional Weighted Voting and Linear Algebra

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Abstract

Voting is a fundamentally mathematical concept; the mathematical field of voting theory was pioneered by Donald Saari. In their paper "Voting, the Symmetric Group, and Representation Theory," Daugherty,Eustis, Minton, and Orrison build off Saari's approach by applying representation theory to the system that Saari has developed. In this paper, we reexamine the results of Daugherty, Eustis, Minton, and Orrison by using a more linear algebra based approach. Specifically, we consider ways to "weight" a voting profile, or, in other words, to score a particular ballot in a vote. This question is important from a practical standpoint in considering the effectiveness or objectivity of voting procedures.

We will prove an important result using linear algebra, namely that there are mathematically infinitely ways people can vote that, when coupled with a specific weighting system, could lead to a specific numerical result. However, this particular conclusion is not always applicable to practical voting situations, and therefore we will show what these theoretically infinite ways people could vote actually means in a real world scenario. We also look at the converse of our original question– whether a particular way that people vote and a particular result can be connected via a weighting system. We also look at the real-world reasonability of this question, and some implications that come from it.

While this paper will only consider the mathematics behind this voting theory, the results will certainly be of interest to anyone interested in ensuring that voting accurately represents the interests of the parties involved, as these results will emphasize that the selection of a weight system for a vote is often more important than the actual vote itself.

Contents

1 Introduction and Motivation

Voting is a fundamentally mathematical concept; the mathematical field of voting theory was pioneered by Donald Saari. Building off Saari's work, which looks at a geometric approach to voting theory, the paper [DEMO2009] by Daugherty, Eustis, Minton, and Orrison apply representation theory to the system that Saari has developed. In this paper, we reexamine the results of Daugherty, Eustis, Minton, and Orrison by looking at a more linear algebra based approach, and by recasting some of their theorems in more real-life scenarios.

One of the primary motivations for studying voting processes with a mathematical approach is the fundamental ambiguity that can arise from voting itself. To illustrate this phenomenon, we turn to a classic example from Donal Saari in [Saari1998]. We can imagine a vote by a 15-member department between beverage options – in this case, milk, wine, and beer – for the departmental picnic. The vote is fully ranked – ie, every voter ranks all three candidates in order of preference. Imagine that our vote goes like this:

Count	6	5	4
Preference Schedule	$\begin{pmatrix} milk \\ wine \\ beer \end{pmatrix}$	$\begin{pmatrix} beer \\ wine \\ milk \end{pmatrix}$	$\begin{pmatrix} wine \\ beer \\ milk \end{pmatrix}$.

Now we can make some observations about this vote. First, we can see that milk is the most popular first choice– six people chose milk first, five people chose beer first, and four people chose wine first. However, milk is also the most popular last choice – nine people preferred milk the least, whereas on six people preferred beer the least, and nobody chose wine the least. Should milk win, or lose? We have this ambiguity. Imagine that we decide milk is the winner – and yet, due to an ironic turn of events, milk is unavailable. Then should the department pick beer or wine? More people chose beer as a first choice than wine – and yet, ten people chose wine over beer, whereas only five people chose beer over wine. In a head-to-head, wine would be the winner. Again, it is ambiguous who should win and who should lose. This fundamental ambiguity is what motivates the idea of positional weighted voting procedures – while they do not completely remove the ambiguity shown here, they do allow a systematic way for picking a winner and quantifying this ambiguity.

2 Positional weighted theory basics

We begin our discussion of voting theory by fully defining the process of weighted positional voting. For a fully ranked vote, voters rank candidates from most preferred to least preferred, just like in Saari's example. If we have n candidates, the voting is equivalent to choosing a column vector of size n, where the top entry of the vector is the number of the first choice, the second entry of the vector is the second choice, et cetera. For example, for three candidates, wine, beer, and milk, a fully ranked vote means that each voter picks one of the following vectors:

$$\begin{pmatrix} milk \\ wine \\ beer \end{pmatrix}, \begin{pmatrix} milk \\ beer \\ wine \end{pmatrix}, \begin{pmatrix} wine \\ milk \\ beer \end{pmatrix}, \begin{pmatrix} wine \\ beer \\ milk \end{pmatrix}, \begin{pmatrix} beer \\ milk \\ wine \end{pmatrix}, \begin{pmatrix} beer \\ wine \\ milk \end{pmatrix}. \tag{1}$$

The idea of positional weighted voting, as outlined in [DEMO2009], is to assign a certain number of points for a first place vote, a certain number of points for a second place vote, et cetera. For example, if we assign one point for a first place vote, every time someone votes for a candidate in first place, that candidate receives one point. At the end of the vote, the candidate with the most points is the winner.

There are many reasonable ways to set up a weighting system. For example a voting system could award one point to first place and zero points to all other places, which would be a "winner-take-all" type system of voting. On the other hand, a weighting system could also give one point to all places except last place, which would be a "consensus" way of voting. A more descending-values type vote might assign two points to first place, one point to second place, and no points to third place. There are infinite variations of the different weighting systems.

Definition 1. A weighting vector, w, is the column vector that defines the weighting system of a vote such that the j^{th} column represents the number of points that the j^{th} place receives.

The first element of the weighting vector is the weight for first place, and the second element of the weighting vector is the weight for second place, et cetera. For size $n = 3$, if we wanted to give one point for a first place, zero points for a second place, and negative one points for last place – note that we are free to use negative values for point counts – we would use the weighting vector:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now, observe that all the vectors that the voters are choosing between are permutations of the original list of candidates. Again, if we're voting between milk, wine, and beer, then voting is the same as choosing one of the permutations of the vector $\begin{pmatrix} milk \\ wine \\ beer \end{pmatrix}$. We now fix an order to the permutations (in the same way you fix an order for an ordered basis); for example, our order for $n = 3$ could be (1), (23),(12),(132),(123),(13). Given this order, the order of our actual vectors is:

$$\begin{pmatrix} milk \\ wine \\ beer \end{pmatrix}, \begin{pmatrix} milk \\ beer \\ wine \end{pmatrix}, \begin{pmatrix} wine \\ milk \\ beer \end{pmatrix}, \begin{pmatrix} wine \\ beer \\ milk \end{pmatrix}, \begin{pmatrix} beer \\ milk \\ wine \end{pmatrix}, \begin{pmatrix} beer \\ wine \\ milk \end{pmatrix}.$$

Definition 2. A profile vector, p, is the column vector that encodes how many people voted for each preference, in relation to the order of the preferences that we've fixed. The j^{th} row represents the number of people that voted for the j^{th} preference in the fixed order.

The first element of the profile vector is the number of people that voted for the first preference schedule, the second element of the profile vector is the number of people that voted for the second preference, et cetera. Continuing with our example of a vote between milk, wine, and beer, if we had 10 voters, imagine we have the profile:

$$p = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}.$$

Then the vote went like this:

Count	3	2	4	1
Preference Schedule	$\begin{pmatrix} milk \\ wine \\ beer \end{pmatrix}$	$\begin{pmatrix} milk \\ beer \\ wine \end{pmatrix}$	$\begin{pmatrix} beer \\ milk \\ wine \end{pmatrix}$	$\begin{pmatrix} beer \\ wine \\ milk \end{pmatrix}$

Definition 3. From the weighting vector we generate the weighting matrix, T_w, such that the rows of T_w correspond to the different candidates and the columns of T_w correspond to our orderings of the

preferences; T_w is then filled such that the i, j element is the number of points that candidate i receives under the j^{th} preference.

The matrix first element of the first row of T_w will be the number of points that our first candidate receives under the first preference we've defined, aka the element of the weighting vector corresponding to the first candidate's position in that preference.

For our candidates milk, wine, and beer, we define the first row to correspond to milk, the second row to correspond to wine, and the third row to correspond to beer. We define the columns to correspond the preferences as we have listed them above. If, for now, we use the arbitrary weighting vector:

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

then we generate T_w as follows:

$$T_w \;=\; \begin{array}{rc} & \begin{array}{cccccc} m & m & w & w & b & b \\ w & b & m & b & m & w \\ b & w & b & m & w & m \end{array} \\ \begin{array}{r} \text{milk} \\ \text{wine} \\ \text{beer} \end{array} & \begin{pmatrix} w_1 & w_1 & w_2 & w_3 & w_2 & w_3 \\ w_2 & w_3 & w_1 & w_1 & w_3 & w_2 \\ w_3 & w_2 & w_3 & w_2 & w_1 & w_1 \end{pmatrix} \end{array}.$$

We can now notice that the weighting matrix T_w is actually the permutations of the weighting vector in the same order as the fixed permutations of the preferences.

Definition 4. The results vector, r, is the column vector where the j^{th} element corresponds to the points awarded to the j^{th}, as ordered in the rows of T_w, candidate after a vote takes place.

If we conduct a vote between milk, wine, and beer, and find that milk has received ten points, wine has received three points, and beer has received seven points, then our results vector is:

$$r = \begin{pmatrix} 10 \\ 3 \\ 7 \end{pmatrix}.$$

This now brings us to the fundamental equation of positional vector voting:

$$T_w p = r. \tag{2}$$

What this tells us is that multiplying our weighting matrix by our profile vector gives us the results vector.

For example, if we use the weighting vector

$$w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

we generate our weighting matrix:

$$T_w = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

And if we have the profile vector

$$p = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix},$$

then we run our fundamental equation $T_w p = r$:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}.$$

This means that milk has scored four points, wine has scored negative six points, and beer has scored two points, making milk the winner.

Example 2.1. While in some cases no matter what weighting system is used there's a clear winner, there are also many cases where selecting a weighting system determines a winner in and of itself. Here is an example of one such paradox.

If we have a vote between four candidates, which we index one through four, where voters are asked to rank these candidates from most preferred to least preferred, this is equivalent to asking our voters to choose between the 24 permutations below, which we fix in this order:

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}1\\2\\4\\3\end{pmatrix}\begin{pmatrix}1\\3\\2\\4\end{pmatrix}\begin{pmatrix}1\\3\\4\\2\end{pmatrix}\begin{pmatrix}1\\4\\2\\3\end{pmatrix}\begin{pmatrix}1\\4\\3\\2\end{pmatrix}\begin{pmatrix}2\\1\\3\\4\end{pmatrix}\begin{pmatrix}2\\1\\4\\3\end{pmatrix}\begin{pmatrix}2\\3\\1\\4\end{pmatrix}\begin{pmatrix}2\\3\\4\\1\end{pmatrix}\begin{pmatrix}2\\4\\1\\3\end{pmatrix}\begin{pmatrix}2\\4\\3\\1\end{pmatrix}$$

$$\begin{pmatrix}3\\1\\2\\4\end{pmatrix}\begin{pmatrix}3\\1\\4\\2\end{pmatrix}\begin{pmatrix}3\\2\\1\\4\end{pmatrix}\begin{pmatrix}3\\2\\4\\1\end{pmatrix}\begin{pmatrix}3\\4\\1\\2\end{pmatrix}\begin{pmatrix}3\\4\\2\\1\end{pmatrix}\begin{pmatrix}4\\1\\2\\3\end{pmatrix}\begin{pmatrix}4\\1\\3\\2\end{pmatrix}\begin{pmatrix}4\\2\\1\\3\end{pmatrix}\begin{pmatrix}4\\2\\3\\1\end{pmatrix}\begin{pmatrix}4\\3\\1\\2\end{pmatrix}\begin{pmatrix}4\\3\\2\\1\end{pmatrix}.$$

Next we must determine our weighting system. We can define, as a basic, beginning step, first place to receive one point, and last place to receive zero points, and used variables s and t for the second and third places.Our weighting vector is defined as:

$$w = \begin{pmatrix}1 & s & t & 0\end{pmatrix}^T.$$

From, w, we can derive T_w, the permutations of the weighting vector in the same order as the permutations of the candidates in the order we've fixed:

$$T_w = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & s & s & t & 0 & t & 0 & s & s & t & 0 & t & 0 & s & s & t & 0 & t & 0\\
s & s & t & 0 & t & 0 & 1 & 1 & 1 & 1 & 1 & 1 & t & 0 & s & s & 0 & t & t & 0 & s & s & 0 & t\\
t & 0 & s & s & 0 & t & t & 0 & s & s & 0 & t & 1 & 1 & 1 & 1 & 1 & 1 & 0 & t & 0 & t & s & s\\
0 & t & 0 & t & s & s & 0 & t & 0 & t & s & s & 0 & t & 0 & t & s & s & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.$$

For a given profile p, encoded as a column vector such that the entry in column i corresponds to the number of people that voted for the voting vector i, $T_w(p) = r$, where r is the results vector where the score in the i row corresponds to the total points assigned to candidate i.

Let $p = (6\ 8\ 2\ 5\ 0\ 0\ 2\ 0\ 6\ 0\ 2\ 4\ 4\ 3\ 3\ 2\ 1\ 2\ 0\ 3\ 6\ 8\ 5\ 2)^T$. Then apply our fundamental equation of vector voting, $r = T_w(p)$, to obtain

$$r = \begin{pmatrix} 21 + 12s + 23t \\ 14 + 33s + 10t \\ 15 + 20s + 23t \\ 24 + 9s + 18t \end{pmatrix}.$$

If we go with a winner-take-all vote, we set $s, t = 0$, and $r = \begin{pmatrix} 21 \\ 14 \\ 15 \\ 24 \end{pmatrix}$, candidate four is the winner. If we go with a consensus system, we set $r = s = 1$, and $r = \begin{pmatrix} 56 \\ 57 \\ 58 \\ 51 \end{pmatrix}$, although candidate four won using the first weighting system, he now comes in last, and candidate three is the winner. We could also allow points for the first two places on the ballot, so that $s = 1, t = 0$. In that case, $r = \begin{pmatrix} 33 \\ 47 \\ 35 \\ 33 \end{pmatrix}$, in which case candidate two is the clear winner. Finally, we could assign second and third place points, but less than first place; for example, we could arbitrarily set $s = t = .5$, and $r = \begin{pmatrix} 38.5 \\ 35.5 \\ 36.5 \\ 37.5 \end{pmatrix}$, in which case, candidate one would be the winner. As shown, given a fairly simple set of candidates and profile, with four different reasonable weighting systems you can get four different winners. Positional weighted voting doesn't eliminate the ambiguity in a vote– however, picking a weighting system allows us to select a winner.

3 Results regarding the weighting vectors

We will be viewing T_w as a linear transformation of vector spaces. Since $\mathbb{N}^n$ is not a vector space, our results space and profile space must be $\mathbb{Q}^n$. In [DEMO2009] Dougherty, Eustace, Minton, and Orrison make an important observation. Namely, they show that since the results space can be decomposed into a weighted space (the vector space spanned by the all-ones vector), direct summed with a sum-zero space (which includes all vectors such that if you add all the elements of the vector together you get zero). Intuitively, we can see that adding any multiple of the all-ones vector to a result will not change the order of the candidates at all. What this means is that we can restrict the results space to sum-zero vectors with no loss of information about the election itself. Since the results space lives in the span of the column space of T_w, and each column of T_w will have the same sum (since they are permutations of the same vector), this means we can restrict our weighting vectors to sum-zero vectors with no loss of information as well. From this point on, when discussing weighting vectors and results vectors, it will be assumed that these vectors are sum-zero.

This brings us to one of the major results of [DEMO2009].

Theorem 3.1. Let n ≥ 2, and let $\lambda = (\lambda_1, ..., \lambda_m)$ be a partition of n. Suppose that $\mathbf{w_1}, ..., \mathbf{w_k}$ form a linearly independent set of weighting vectors in $\mathbb{Q}^m$ such that $\bar{\mathbf{w_1}}, ..., \bar{\mathbf{w_k}}$ are sum-zero vectors. If $\mathbf{r_1}, ..., \mathbf{r_k}$ are any sum-zero results vectors in $\mathbb{Q}^n$, then there exist infinitely many profiles $\mathbf{p} \in M^\lambda$ such that $T_{w_i}(\mathbf{p}) = \mathbf{r}_i$ for all i such that $1 \leq i \leq k$.

For a fully-ranked vote, what this means is that given any sum-zero results vectors and any linearly independent sum-zero weighting vectors, there are actually infinitely many profiles that will simultaneously connect the weighting vectors to the results via our fundamental equation $T_w p = r$.

Example 3.1. To see how Theorem 3.1 works, let's look at an example of what it means. Define our weighting vectors to be

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \text{ and } w_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Now we define our results vectors as

$$r_1 = \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix}, \text{ and } r_2 = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}.$$

Theorem 3.1 says that we should be able to find infinitely many profiles such that we can connect these weighting vectors to these results. Notice that these weighting vectors are linearly independent and sum-zero, and that these results are sum-zero, which are prerequisites for the applicability of Theorem 3.1. Now, we can generate our two weighting matrices from these weighting vectors such that we get the following two equations:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix} p = \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 1 & 1 & 1 & -2 & 1 & -2 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ -2 & 1 & -2 & 1 & 1 & 1 \end{pmatrix} p = \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}.$$

We can solve this system easily using linear algebra by combining these two matrices into a single matrix, and these two results vectors into a single vector, and then solving using row reduction for the profile. When we do this, we get that:

$$p = \begin{pmatrix} p_5 + 1 \\ p_6 + 11 \\ p_6 + 6 \\ p_5 + 7 \\ p_5 \\ p_6 \end{pmatrix}.$$

We see that this profile has two free variables; this means that there are infinitely many profiles which would fit this criteria. Thus, for these specific weighting vectors and results, we see that Theorem 3.1 holds.

Dougherty et al prove their theorem using an argument from representation theory. We set out to prove the same theorem using linear algebra tools. We begin with a smaller version of the theorem:

Theorem 3.2. Let n=3, and let $\lambda = (\lambda_1, ..., \lambda_m)$ be a partition of n. Suppose that $\mathbf{w_1}, ..., \mathbf{w_k}$ form a linearly independent set of weighting vectors in $\mathbb{Q}^m$ such that $\bar{\mathbf{w_1}}, ..., \bar{\mathbf{w_k}}$ are sum-zero vectors. If $\mathbf{r_1}, ..., \mathbf{r_k}$ are any sum-zero results vectors in $\mathbb{Q}^n$, then there exist infinitely many profiles $\mathbf{p} \in M^\lambda$ such that $T_{w_i}(\mathbf{p}) = \mathbf{r}_i$ for all i such that $1 \leq i \leq k$.

Proof. Since if n=3 we have two linearly independent weighting vectors

$$w_1 = \begin{pmatrix} a \\ b \\ -(a+b) \end{pmatrix}, \text{ and } w_2 = \begin{pmatrix} c \\ d \\ -(c+d) \end{pmatrix}.$$

And let our arbitrary results vectors be

$$R_1 = \begin{pmatrix} r_1 \\ r_2 \\ -r_1 - r_2 \end{pmatrix}, \text{ and } R_2 = \begin{pmatrix} r_3 \\ r_4 \\ -r_3 - r_4 \end{pmatrix}.$$

We then generate our two T_w matrices based on these weighting vectors:

$$T_{w_1} = \begin{pmatrix} a & a & b & -a-b & b & -a-b \\ b & -a-b & a & a & -a-b & b \\ -a-b & b & -a-b & b & a & a \end{pmatrix},$$

$$T_{w_2} = \begin{pmatrix} c & c & d & -c-d & d & -c-d \\ d & -c-d & c & c & -c-d & d \\ -c-d & d & -c-d & d & c & c \end{pmatrix}.$$

We must show there are infinite profiles

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$

that solve the equations:

$$T_{w_1} p = R_1, \text{ and}$$

$$T_{w_2} p = R_2.$$

We can combine these into a single equation by defining T_w to be T_{w_1} on top of T_{w_2} and R to be R_1 on top of R_2. Then we have

$$T_w p = R,$$

or, more explicitly, we have

$$\begin{pmatrix} a & a & b & -a-b & b & -a-b \\ b & -a-b & a & a & -a-b & b \\ -a-b & b & -a-b & b & a & a \\ c & c & d & -c-d & d & -c-d \\ d & -c-d & c & c & -c-d & d \\ -c-d & d & -c-d & d & c & c \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ -r_1-r_2 \\ r_3 \\ r_4 \\ -r_3-r_4 \end{pmatrix}.$$

Note that, for both T_w and R, the third and sixth rows are each the negation of the two rows above it, and therefore unnecessary to this system of equations. We can therefore cancel them out. This leaves us with four equations to solve for six unknowns, meaning that there are either infinite solutions or none.

We now assume to the contrary, ie that there are no solutions to this system of equations. For this to be the case, the remaining rows must be linearly dependent. Therefore, for some constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, not all zero, we have the following system of equations:

$$\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d = 0 \tag{3}$$

$$\lambda_1 a + \lambda_2(-a-b) + \lambda_3 c + \lambda_4(-c-d) = 0 \tag{4}$$

$$\lambda_1 b + \lambda_2 a + \lambda_3 d + \lambda_4 c = 0 \tag{5}$$

$$\lambda_1(-a-b) + \lambda_2 a + \lambda_3(-c-d) + \lambda_4 c = 0 \tag{6}$$

$$\lambda_1 b + \lambda_2(-a-b) + \lambda_3 d + \lambda_4(-c-d) = 0 \tag{7}$$

$$\lambda_1(-a-b) + \lambda_2 b + \lambda_3(-c-d) + \lambda_4 d = 0 \tag{8}$$

Note that $2\lambda_1 - \lambda_2$ and $2\lambda_2 - \lambda_1$ cannot both be zero, because as per the previous system of equations this would imply $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are all zero, which is not possible.

If $2\lambda_1 - \lambda_2 \neq 0$ we can combine equations (3) and (4) to set:

$$\begin{aligned} (2\lambda_1 - \lambda_2)a &= -(2\lambda_3 - \lambda_4)c, \\ a &= \frac{-(2\lambda_3 - \lambda_4)}{2\lambda_1 - \lambda_2}c. \end{aligned}$$

Similarly combining equations (5) and (7) yields $b = \frac{-(2\lambda_3 - \lambda_4)}{2\lambda_1 - \lambda_2}d$, but then w_1 divides w_2 so we no longer have linear dependence between our weighting vectors. $2\lambda_1 - \lambda_2 = 0$.

Therefore, $2\lambda_2 - \lambda_1 \neq 0$. We can combine equations (3) and (8) to set:

$$\begin{aligned} (-\lambda_1 + 2\lambda_2)b &= -(-\lambda_3 + 2\lambda_4)d \\ b &= \frac{-(2\lambda_4 - \lambda_3)}{2\lambda_2 - \lambda_1} d. \end{aligned}$$

Similarly combining equations (5) and (6) gives us $a = \frac{-(2\lambda_4 - \lambda_3)}{2\lambda_2 - \lambda_1} c$. This means, again, that w_1 divides w_2, and therefore our weighting vectors are linearly dependent, which contradicts our assumption. Therefore, there must be a solution to this problem, and therefore there are infinite solutions. $\square$

4 Proof by contradiction for arbitrary n

Next we generalize our results to prove Theorem 3.1.

Proof. This proof will follow the same steps as the proof for n=3, but for an arbitrary size of weighting vector. Since our original theorem works for any linearly independent set of weighting vectors, and for size n weighting vectors there can be up to $n-1$ total linearly independent weighting vectors, if we prove that the theorem holds for $n-1$ linearly independent weighting vectors it will hold for any number of weighting vectors. Our weighting vectors $\{W_1, W_2, ..., W_{n-1}\}$ are:

$$\begin{pmatrix} w_{1,1} \\ w_{1,2} \\ w_{1,3} \\ ... \\ w_{1,n-1} \\ -(w_{1,1} + ... + w_{1,n-1}) \end{pmatrix}, \begin{pmatrix} w_{2,1} \\ w_{2,2} \\ w_{2,3} \\ ... \\ w_{2,n-1} \\ -(w_{2,1} + .. + w_{2,n-1}) \end{pmatrix}, ..., \begin{pmatrix} w_{n-1,1} \\ w_{n-1,2} \\ w_{n-1,3} \\ ... \\ w_{n-1,n-1} \\ -(w_{n-1,1} + ... + w_{n-1,n-1}) \end{pmatrix}.$$

We form our matrix T_w, where the columns represent all the permutations of the individual weighting vectors (not the permutations of the whole column), and our arbitrary results vectors, and get our equation:

$$\begin{pmatrix} w_{1,1} & ..... \\ w_{1,2} & ..... \\ w_{1,3} & ..... \\ ... & ..... \\ w_{1,n-1} & ..... \\ -(w_{1,1}+..+w_{1,n-1}) & ..... \\ w_{2,1} & ..... \\ w_{2,2} & ..... \\ w_{2,3} & ..... \\ ... & ..... \\ w_{n-1,n-1} & ..... \\ -(w_{n-1,1}+..+w_{n-1,n-1}) & ..... \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ .. \\ p_{n!} \end{pmatrix} = \begin{pmatrix} r_{1,1} \\ r_{1,2} \\ .. \\ r_{1,n-1} \\ -(r_{1,1}+r_{1,2}+...) \\ .. \\ r_{n-1,1} \\ -(r_{n-1,n}+r_{n-1,2}+...) \end{pmatrix}.$$

Again, we cancel out every nth row, since, in both the result and T_w this row is the negation of the $n-1$ rows above it. We observe that, after canceling these rows out, we have n^2-2n+1 rows and $n!$ columns. This means that, since $n! > n^2-2n+1$ for $n > 2$, we have that there are either infinite or no solutions. Again, for there to be no solutions, the remaining rows would have to be linearly dependent. Therefore, for a set of constants $\{\lambda_{1,1}, \lambda_{1,2}, .., \lambda_{1,n-1}, ..\lambda_{n-1,n-1}\}$ we have our system of equations for the rows: ($\lambda_{1,1}\cdot$ row 1) + ($\lambda_{1,2}\cdot$ row 2)+...+($\lambda_{n-1,n-1}\cdot$ last row) $= \vec{0}$. The subscript of the λ matches the subscript of the weighting vector in the first column of the row that it's multiplied by. We have a different homogeneous equation for each column of our matrix.

Now select all the equations where the first element of the first weighting vector, ie $w_{1,1}$, is multiplied by λ_1, and the other elements of the weighting vector are shifted down but not rearranged. Since each weighting vector is run through the same set of permutations, for every weighting vector the first element is also fixed in the first position, and the other elements are shifted by the same amount. We've selected the set of equations:

$$\begin{aligned}
\lambda_{i,1}w_{i,1}+\lambda_{i,2}(-w_{i,1}-w_{i,2}-..-w_{i,n-1})+\lambda_{i,3}w_{i,2}+..+\lambda_{i,n-1}w_{i,n-2} &= 0\\
\lambda_{i,1}w_{i,1}+\lambda_{i,2}w_{i,n-1}+\lambda_{i,3}(-w_{i,1}-w_{i,2}-..-w_{i,n-1})+...+\lambda_{i,n-1}w_{i,n-3} &= 0\\
\lambda_{i,1}w_{i,1}+\lambda_{i,2}w_{i,n-2}+\lambda_{i,3}w_{i,n-1}+...+\lambda_{i,n-1}w_{i,n-4} &= 0\\
&...\\
\lambda_{i,1}w_{i,1}+\lambda_{i,2}w_{i,3}+\lambda_{i,3}w_{i,4}+...+\lambda_{i,n-1}(-w_{i,1}-w_{i,2}-..-w_{i,n-1}) &= 0.
\end{aligned}$$

Now, notice that every value multiplied by $\lambda_{i,1}$ is the value $w_{i,1}$, and, since we cycle through the weights with the other values, we get each other value λ multiplied by each element of the weighting vector besides $w_{i,1}$ as well as multiplied by the negation of all the elements of the weighting vector. Therefore, when we combine all these equations, we find that, for each weighting vector:

$$((n-1)\lambda_{i,1}-\lambda_{i,2}-\lambda_{i,3}-..-\lambda_{i,n-1})w_{i,1}=0.$$

So

$$\sum_{i=1}^{n-1}((n-1)\lambda_{i,1}-\lambda_{i,2}-\lambda_{i,3}-..-\lambda_{i,n-1})w_{i,1}=0.$$

We then combine the equations here where the second value of the weighting vector is multiplied by $\lambda_{i,1}$, and the rest of the values are shifted through the other values and not rearranged. By combining these equations, we see:

$$\sum_{i=1}^{n-1}((n-1)\lambda_{i,1}-\lambda_{i,2}-\lambda_{i,3}-..-\lambda_{i,n-1})w_{i,2}=0.$$

We continually do this for all the elements, finding that, for the kth value of the weighting vector,

$$\sum_{i=1}^{n-1}((n-1)\lambda_{i,1}-\lambda_{i,2}-\lambda_{i,3}-..-\lambda_{i,n-1})w_{i,k}=0.$$

This means that:

$$\sum_{i=1}^{n-1}((n-1)\lambda_{i,1}-\lambda_{i,2}-\lambda_{i,3}-..-\lambda_{i,n-1})W_{i}=0.$$

If we repeat the same process, but fix the vectors multiplying by $\lambda_{i,2}$ instead of $\lambda_{i,1}$, we get:

$$\sum_{i=1}^{n-1}((n-1)\lambda_{i,2}-\lambda_{i,1}-\lambda_{i,3}-..-\lambda_{i,n-1})W_i=0.$$

We continue repeating this process, getting that, for all values $1 \leq j \leq n-1$

$$\sum_{i=1}^{n-1}((n-1)\lambda_{i,j}-\lambda_{i,1}-\lambda_{i,2}-..-\lambda_{i,n-1})W_i=0.$$

But since the weighting vectors must be linearly independent, each coefficient $((n-1)\lambda_{i,j}-\lambda_{i,1}-\lambda_{i,2}-..-\lambda_{i,n-1})=0$. However, this implies that would mean that each value λ would have to equal 0, which is a contradiction. Therefore, there must be a profile that solves for an arbitrary weighting vector and results vector, which means that there are infinitely many solutions, thus proving Theorem 3.1. $\square$

5 Reversal of Theorem 3.1

Now that we've proven Theorem 3.1 – that for any set of linearly independent sum-zero weighting vectors and any set of sum-zero results vectors, there are infinitely many profiles that connect these via our fundamental equation – this leads to a natural reversal of the question: can any profile and results vector be connected by choosing a weighting vector? Currently, we have restricted our considerations to three-candidate elections.

In order to look at this question, we first need to reverse our fundamental equation $T_w p = r$ so that we can examine the weighting vector as an independent variable we can manipulate. We want to generate a matrix T_p from the profile p such that we've reversed our fundamental equation $T_p w = r$.

Definition 5. From the profile p we generate our profile matrix T_p such that the rows of T_p correspond to the candidates and the columns of T_p correspond to the positions, ie 1st, 2nd, 3rd, etc., and the i, j element of T_p corresponds to the number of voters who voted for the i candidate in the j position on their ballot.

The first element of the first row of T_p will be the number of people that voted for the first candidate as their first-place choice.

To see how this will work, using candidates wine, beer, and milk, and the ordering given in Equation

(1), suppose we're given the generalized profile:

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}.$$

Then we can see that (p_1+p_2) people voted for candidate one in first place; (p_3+p_4) voted for candidate two in first place; (p_5+p_6) people voted for candidate three in first place, et cetera. We generate T_p in this way:

$$T_p \;=\; \begin{array}{r} \\ \text{milk} \\ \text{wine} \\ \text{beer} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \text{1st place} & \text{2nd place} & \text{3rd place} \end{array} \\ \begin{pmatrix} p_1+p_2 & p_3+p_5 & p_4+p_6 \\ p_3+p_4 & p_1+p_6 & p_2+p_5 \\ p_5+p_6 & p_2+p_4 & p_1+p_3 \end{pmatrix} \end{array}.$$

Now with this definition of our profile matrix T_p we can define our reversed fundamental equation of positional weighted voting:

$$T_p w = r. \tag{9}$$

Now we can multiply our profile matrix by our weighting vector to get our result.

We also now know that $T_w p = T_p w$. We can illustrate that fact with an example. Here we'll use both a generalized weighting vector and a generalized profile vector:

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} \text{ and } w = \begin{pmatrix} w_1 \\ w_2 \\ -w_1-w_2 \end{pmatrix}.$$

Then, as we've shown, we can generate

$$T_w = \begin{pmatrix} w_1 & w_1 & w_2 & -w_1 - w_2 & w_2 & -w_1 - w_2 \\ w_2 & -w_1 - w_2 & w_1 & w_1 & -w_1 - w_2 & w_2 \\ -w_1 - w_2 & w_2 & -w_1 - w_2 & w_2 & w_1 & w_1 \end{pmatrix},$$

$$T_p = \begin{pmatrix} p_1 + p_2 & p_3 + p_5 & p_4 + p_6 \\ p_3 + p_4 & p_1 + p_6 & p_2 + p_5 \\ p_5 + p_6 & p_2 + p_4 & p_1 + p_3 \end{pmatrix}.$$

$$\begin{aligned} T_w p &= \begin{pmatrix} w_1 & w_1 & w_2 & -w_1 - w_2 & w_2 & -w_1 - w_2 \\ w_2 & -w_1 - w_2 & w_1 & w_1 & -w_1 - w_2 & w_2 \\ -w_1 - w_2 & w_2 & -w_1 - w_2 & w_2 & w_1 & w_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} \\ &= \begin{pmatrix} (p_1 + p_2 - p_4 - p_6)w_1 + (p_3 + p_5 - p_4 - p_6)w_2 \\ (p_3 + p_4 - p_2 - p_5)w_1 + (p_1 + p_6 - p_2 - p_5)w_2 \\ (p_5 + p_6 - p_1 - p_3)w_1 + (p_2 + p_4 - p_1 - p_3)w_2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} T_p w &= \begin{pmatrix} p_1 + p_2 & p_3 + p_5 & p_4 + p_6 \\ p_3 + p_4 & p_1 + p_6 & p_2 + p_5 \\ p_5 + p_6 & p_2 + p_4 & p_1 + p_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ -w_1 - w_2 \end{pmatrix} \\ &= \begin{pmatrix} (p_1 + p_2 - p_4 - p_6)w_1 + (p_3 + p_5 - p_4 - p_6)w_2 \\ (p_3 + p_4 - p_2 - p_5)w_1 + (p_1 + p_6 - p_2 - p_5)w_2 \\ (p_5 + p_6 - p_1 - p_3)w_1 + (p_2 + p_4 - p_1 - p_3)w_2 \end{pmatrix}. \end{aligned}$$

From this we can see that by generating either our profile matrix or our weighting matrix and then applying either Equation (2) or Equation (9) leads to the same results vector outcome.

To examine our reversal of Theorem 3.1 we're going to simplify the mechanics of Equation (9). We look at the equation:

$$T_p w = \begin{pmatrix} p_1+p_2 & p_3+p_5 & p_4+p_6 \\ p_3+p_4 & p_1+p_6 & p_2+p_5 \\ p_5+p_6 & p_2+p_4 & p_1+p_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ -w_1-w_2 \end{pmatrix}.$$

Notice that we can simplify this equation by eliminating the last column of T_p, the last row of w, and adding the negation of the third column of T_p to every other column of T_p. We now verify that these are equivalent:

$$\begin{aligned} & \begin{pmatrix} p_1+p_2-p_4-p_6 & p_3+p_5-p_4-p_6 \\ p_3+p_4-p_2-p_5 & p_1+p_6-p_2-p_5 \\ p_5+p_6-p_1-p_3 & p_2+p_4-p_1-p_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ = & \begin{pmatrix} (p_1+p_2-p_4-p_6)w_1+(p_3+p_5-p_4-p_6)w_2 \\ (p_3+p_4-p_2-p_5)w_1+(p_1+p_6-p_2-p_5)w_2 \\ (p_5+p_6-p_1-p_3)w_1+(p_2+p_4-p_1-p_3)w_2 \end{pmatrix} \\ = & \ T_p w. \end{aligned}$$

We see that these are equivalent. As a reminder, our reversed question is: if we're given a profile, can we find a weighting vector to ensure a given result. Now we see that this is equivalent to asking: for a given result, is that result in the column space of our matrix:

$$\begin{pmatrix} p_1+p_2-p_4-p_6 & p_3+p_5-p_4-p_6 \\ p_3+p_4-p_2-p_5 & p_1+p_6-p_2-p_5 \\ p_5+p_6-p_1-p_3 & p_2+p_4-p_1-p_3 \end{pmatrix}.$$

If our result is in the column space of this matrix, we see that we can then generate a weighting vector to ensure our result, given that our weighting vector just takes a linear combination of the columns of this matrix. We see then that, since our columns of this matrix are sum-zero, and our result is sum-zero, if the two columns of this matrix are linearly independent then they form a basis for the sum-zero results space, and thus the result will be in the column space. Therefore, we can get our result as long as columns

$$\begin{pmatrix} p_1+p_2-p_4-p_6 \\ p_3+p_4-p_2-p_5 \\ p_5+p_6-p_1-p_3 \end{pmatrix} \text{ and } \begin{pmatrix} p_3+p_5-p_4-p_6 \\ p_1+p_6-p_2-p_5 \\ p_2+p_4-p_1-p_3 \end{pmatrix}$$

are linearly independent. This will be true exactly when:

$$-{p_6}^2-{p_2}^2-p_5p_1-p_5p_4-{p_3}^2+p_3p_6+{p_4}^2+p_2p_3+{p_5}^2+{p_1}^2-p_1p_4+p_6p_2 \quad \neq \quad 0. \tag{10}$$

We now observe three different types of profiles: profiles which only yield the zero result (ie both columns of our matrix are zero), profiles which can yield only a result that is a multiple of a single vector (ie the columns of our matrix are linearly dependent), and profiles which can yield any result (ie the columns of our matrix are linearly independent). Now we'll show that these three categories exist, and what profile vectors in them look like.

For a profile to be in our first category, only yielding the zero result, both columns of our matrix must equal zero:

$$\begin{pmatrix} p_1+p_2-p_4-p_6 & p_3+p_5-p_4-p_6 \\ p_3+p_4-p_2-p_5 & p_1+p_6-p_2-p_5 \\ p_5+p_6-p_1-p_3 & p_2+p_4-p_1-p_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that we have the system of equations:

$$p_1+p_2-p_4-p_6=0$$

$$p_3+p_5-p_4-p_6=0$$

$$p_3+p_4-p_2-p_5=0$$

$$p_1+p_6-p_2-p_5=0$$

$$p_5+p_6-p_1-p_3=0$$

$$p_2+p_4-p_1-p_3=0.$$

Solving for this system of equations gives us:

$$p_1=p_4=p_5$$

$$p_2 = p_3 = p_6.$$

So we see that the profiles that can only give us the zero result are of the form:

$$p = \begin{pmatrix} a \\ b \\ b \\ a \\ a \\ b \end{pmatrix}.$$

Example 5.1. We'll now look at an example of a profile of this form. Imagine that we're given the profile:

$$p = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

Now, we can look at a table for how many people voted for each candidate position:

	1st place	2nd place	3rd place
Milk	4	4	4
Wine	4	4	4
Beer	4	4	4

We can see that no matter how we weight first, second, and third places, the candidates have to be tied; and, since we're working with sum-zero results vectors, this means that the result must be the zero vector. Just to confirm, we can multiply this vector by our generalized weighting matrix:

$$T_w p = \begin{pmatrix} w_1 & w_1 & w_2 & -w_1 - w_2 & w_2 & -w_1 - w_2 \\ w_2 & -w_1 - w_2 & w_1 & w_1 & -w_1 - w_2 & w_2 \\ -w_1 - w_2 & w_2 & -w_1 - w_2 & w_2 & w_1 & w_1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This profile does indeed only give us the zero result, and any vector of the form we've shown can also only ever give us the zero result.

We can also look at vectors for which we can only find results that are multiples of a certain vector. We know we cannot generate any result from a given profile if the following equation holds:

$-{p_6}^2 - {p_2}^2 - p_5p_1 - p_5p_4 - {p_3}^2 + p_3p_6 + {p_4}^2 + p_2p_3 + {p_5}^2 + {p_1}^2 - p_1p_4 + p_6p_2 = 0.$

Solving this equation, we find that our solutions are of the form:

$$p_1 = 1/2\,p_5 + 1/2\,p_4 \pm 1/2\,\sqrt{-3\,{p_5}^2 + 6\,p_5p_4 - 3\,{p_4}^2 + 4\,{p_6}^2 + 4\,{p_2}^2 + 4\,{p_3}^2 - 4\,p_3p_6 - 4\,p_2p_3 - 4\,p_6p_2}.$$

Our profiles for which we cannot fully determine the point count of a result by changing the weighting vector are the profiles of the form:

$$p = \begin{pmatrix} 1/2\,p_5 + 1/2\,p_4 \pm 1/2\,\sqrt{-3\,{p_5}^2 + 6\,p_5p_4 - 3\,{p_4}^2 + 4\,{p_6}^2 + 4\,{p_2}^2 + 4\,{p_3}^2 - 4\,p_3p_6 - 4\,p_2p_3 - 4\,p_6p_2} \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}.$$

Our total profile space is six-dimensional; inside it is a five-dimensional subspace that we cannot fully determine the results; and inside that is a two dimensional subspace whose results will only be the zero subspace.

Example 5.2. Just to look at an example of a profile for which we CAN determine the full point count of our result by changing the weighting vector. Imagine we're given the profile

$$p = \begin{pmatrix} 10 \\ 4 \\ 5 \\ 2 \\ 4 \\ 6 \end{pmatrix} \text{ and}$$

a generalized result

$$r = \begin{pmatrix} r_1 \\ r_2 \\ -r_1 - r_2 \end{pmatrix}.$$

Then we can solve by generating our matrix T_p, and then row reducing that matrix with this result to find the values of our weighting vector. When we do that, we find that our solution is:

$$w = \begin{bmatrix} \frac{8}{49} r_1 - \frac{1}{49} r_2 \\ \frac{6}{49} r_2 + \frac{1}{49} r_1 \\ -\frac{9}{49} r_1 - \frac{5}{49} r_2 \end{bmatrix}.$$

We see then that for any result, we can then generate a specific weighting vector to match our profile to our result via Equation (9).

Note that, since our total profile vector space is of a greater dimension than our vectors which we cannot determine the result from, the percentage of nice profiles, for which we can determine the full result, increases at a higher rate than not nice profiles; therefore, as the number of voters goes to infinity, the percentage of profiles which you can, as in the example previous, choose any result by determining a weighting vector goes to 100. This actually seems to correspond fairly tightly with Dougherty et al's Theorem 3.1, since the profiles we can find under Theorem 3.1 can be of any size, and, therefore, in some sense, are allowed to be infinitely big, implying an infinite amount of voters. By restricting the voter count, Theorem 3.1 would not necessarily hold.

6 Reality?

We now have a question about our parallel results from Theorem 3.1 and its converse question. As per Theorem 3.1, we know that we can find infinitely many profiles to connect a set of linearly independent weighting vectors to any set of results. However, Theorem 3.1 actually only guarantees that the infinitely many profiles that we can find are vectors of rational numbers. We know that the profile represents the number of people that vote for a particular preference. This means that for a profile to be possible in the real world, it should be a vector of integers. Not all of the profiles we are guaranteed mathematically under Theorem 3.1 are actually realistic.

Likewise, we also examine the realism of answer to the converse of Theorem 3.1. We've shown that, for some profiles (most profiles, as the number of voters increases), you can find a weighting vector to connect a profile vector to a results vector. However, the weighting vector corresponds to the weights assigned to first place, second place, third place, et cetera. You would never, in a reasonable vote, assign more points to last place than to first. We have this criteria of reasonable weighting vectors such that the weighting vector's values must be in descending order.

Definition 6. For three candidates, a reasonable weighting vector is one such that

$$w_1 \geq w_2 \geq w_3.$$

Like we saw with Theorem 3.1, only a subset of the weighting vectors we can find under this reversal could actually be used in the real world.

7 The reasonable results space

To illustrate an example of unreasonable and reasonable weighting vectors, we can revisit our example of a profile from which you can generate any results. Recall:

$$p = \begin{pmatrix} 10 \\ 4 \\ 5 \\ 2 \\ 4 \\ 6 \end{pmatrix}, w = \begin{bmatrix} \frac{8}{49}r_1 - \frac{1}{49}r_2 \\ \frac{6}{49}r_2 + \frac{1}{49}r_1 \\ -\frac{9}{49}r_1 - \frac{5}{49}r_2 \end{bmatrix}.$$

And we know that from this profile, the total possible results space includes all sum-zero results vectors, as we can find a weighting vector for any given result.

However, imagine we want to get the result

$$r = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

Then, as per our equation of the weighting vector:

$$w = \begin{bmatrix} -\frac{8}{49} + \frac{1}{49} \\ \\ -\frac{6}{49} - \frac{1}{49} \\ \\ \frac{9}{49} + \frac{5}{49} \end{bmatrix}.$$

From this, we can see very clearly that w_3 is greater than w_1 and w_2. This means that under this weighting scheme, we'd be giving more points to last place than to first or second – that system certainly isn't reasonable in a real world scenario.

In order to look solely at reasonable results, now we want to restrict our search to reasonable weighting vectors. We can just impose the following inequality:

$$w_1 \geq w_2 \geq w_3.$$

Which is, given this weighting vector:

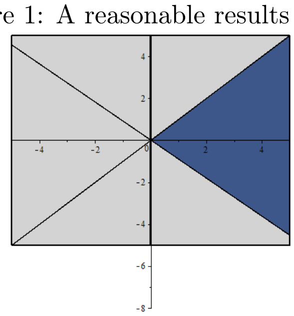
$$\frac{8}{49} r_1 - \frac{1}{49} r_2 \geq \frac{6}{49} r_2 + \frac{1}{49} r_1 \geq -\frac{9}{49} r_1 - \frac{5}{49} r_2.$$

Remember that r_1 is the total points awarded to Candidate 1, r_2 is the total points awarded to Candidate 2, and the total points awarded to Candidate 3 must be $-r_1 - r_2$. Now that we have this inequality, this forms a reasonable results space.

Definition 7. The reasonable results space is the space spanned by the results from all possible reasonable weighting vectors.

We can graph the inequality we've found in our current example, setting r_1 on the x-axis and r_2 on the y-axis.

Figure 1: A reasonable results space



For our nice profiles we can find this reasonable results space and look at it graphically in this way.

8 An observation looking at the results space

If we are looking at the results space graphically, for our three candidates, milk, wine, and beer, such that r_1 (equal to the total points awarded to milk) is on the x-axis and r_2 (equal to the total points awarded to wine) is on the y-axis, and r_3 (equal to the total points awarded to beer) is known to be $-r_1 - r_2$, then we can look at the portions of a graph such that each candidate is the winner; ie, there is a portion of the graph that our first candidate, milk, is the winner; there's a portion of the graph such that wine, our second candidate, is the winner; and there's a portion of the graph such that beer is the winner. The area that would correspond to milk winning would be the area that solves the inequalities:

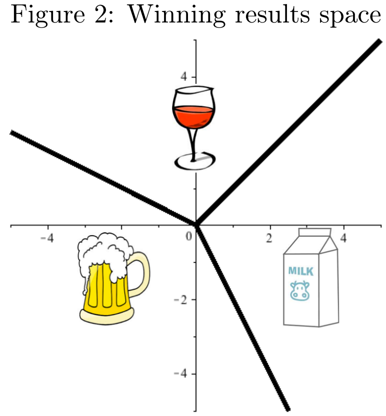
$$r_1 > r_2 \text{ and } r_1 > (-r_1 - r_2).$$

And the area that solves this is of the form:

$$x > 0, x > y \text{ and } 2x > -y.$$

And likewise, we can solve for the area corresponding to each of the other candidates winning. The following graph shows the results space divided up into it's winning spaces (along the lines of the graph we have ties between the candidates).

Figure 2: Winning results space



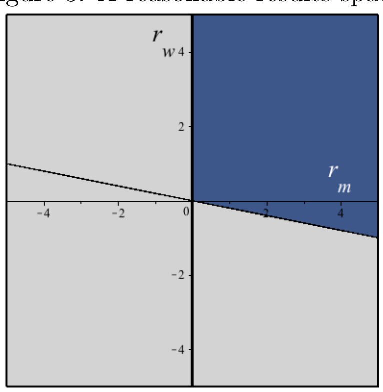
9 A new voting technique?

This seems to suggest to us a new technique of voting. One of the principle causes of ambiguity in positional weighted voting is in the choosing of the weighting vector, out of many equally reasonable weighting vectors; and, as we've shown, this choice of weighting vector can change the results of a vote dramatically. But what looking at the reasonable results suggests to us is that we can vote in this way without actually picking a specific weighting vector; rather, from any profile we can generate its reasonable results space, and then calculate the proportions of that results space that lie in each candidate's winning space. Then the candidate with the highest percent is the winner of the election. Instead of choosing a specific weighting vector, this method would be looking at all possible reasonable weighting vectors.

As per our previous example, we can compare our graph of our reasonable results space with our candidates' winning regions, we see that clearly, candidate one would win.

Example 9.1. As another example, imagine we're given the following reasonable results space for a milk/wine/beer vote:

Figure 3: A reasonable results space



Then comparing this graph to our winning regions, we see that most of the reasonable results space goes to milk; the second most goes to wine; and none of the reasonable results have beer winning. Thus, by this new method of voting, we have milk winning, followed by wine, followed by beer. Notice that, if we choose a specific weighting vector, wine could win.

10 Future research

Future research could explore how this potential voting technique lines up with certain established standards for voting procedures– for example, the idea of a Condorcet winner, which is a winner who beats any other individual candidate in a head-to-head preference of voters. We're interested in how and when this system produces Condorcet winners.

We're also interested in looking at ways of representing this system with more than three candidates. Obviously, in the way we've outlined here we need to be able to view a two-dimensional graph of the reasonable results space; with more candidates, however, we would need to look at a higher-dimensional graph. So potentially this procedure could be systematized for more than three candidates in the future.

Furthermore, Theorem 3.1 holds, in part, due to the fact that the profiles allowed under it can be of an approaching infinitely large size. We could consider looking at the application of Theorem 3.1 when we put a restriction on the number of voters. If we put such a restriction, when can we find a profile to connect weighting vectors to results?

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