

ON THE MINUSCULE REPRESENTATION OF TYPES A_n AND D_n

by

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Abstract

In the late nineteenth century, Killing and Cartan discovered and classified all finite-dimensional simple Lie algebras over the complex numbers. Shortly after this, all irreducible representations of these algebras were classified as well. Among these representations, minuscule representations play an important role.

It is known that the minuscule representations of simple Lie algebras are irreducible. My goal is to show the irreducibility of a minuscule representation using only the cycle structures of the Weyl group elements viewed as permutations acting on the set of weights of that representation. In this project we focus on the simple Lie algebras of type A_n and D_n . Using a computer program [Co], we are able to study the cycle structures of these permutations and eliminate possible dimensions of submodules to show the irreducibility of minuscule representations in many cases.

1 Introduction

This project is an attempt to show the irreducibility of minuscule representations of finite dimensional Lie algebras using the cycle structure of Weyl group elements alone as they act as permutations on the set of weights in an orbit. This problem has been previously investigated for specific choices of minuscule weights for certain types of simple Lie algebras.

We know from [CMS] that in type A_n we can always see the irreducibility of the minuscule module of highest weight λ_1 using one cycle structure, namely, the structure of the Coxeter element. This element always yields an $(n + 1)$ -cycle, and thus prevents any non-trivial proper submodules, as the set of weights has size $(n + 1)$. Due to the symmetry of the Dynkin diagram, this result also applies to λ_n .

More recently, we know that this system of showing irreducibility through cycle structure alone fails for many minuscule modules of type B_n , as shown in [CH]. Furthermore, this paper conjectures that there are infinitely many minuscule modules of type B_n for which this approach fails.

This paper investigates the remaining minuscule modules of simple Lie algebras of types A_n and D_n . We found that one can see the irreducibility of all minuscule modules of type A_n for $n = 1, 2, 3, 4, 5, 6, 7$, and 8 , and also we can see the irreducibility of the λ_1, λ_2 , and λ_3 minuscule modules for $n = 9$. Our computer program could not continue for any other weights of A_9 or higher ranks, but we conjecture that one can see the irreducibility of the minuscule weights through cycle structures alone for all type A_n simple Lie algebras. Unfortunately, the number of elements needed to see irreducibility

will increase with rank and as the weights become closer to the center of the Dynkin diagram. For type D_n , we show that one can see the irreducibility of all minuscule weights through cycle structures alone for $n = 4, 5, 6$, and 7 . We could not continue past rank 7 for type D_n , but we conjecture that one can see irreducibility of type D_n for all ranks and modules.

Since Lie algebras are unfamiliar to most undergraduate students, this paper will provide a comprehensive background needed to fully understand the objective of our project, followed by the results of the project.

2 Lie Algebras

To lay the ground work leading into this project, we must first discuss basic definitions and common examples of Lie algebras, their properties, and related structures. This background is essential to understanding the objective of the project.

Background information in the section is developed from chapters 1–5 of an introductory source, [M], and chapters 1–5 of a slightly more advanced source, [EW]. For more comprehensive discussion of these concepts, we direct the reader to chapters 1 and 2 of [Ca] and chapters 1–3 of [H1].

Lie algebras are built on top of vector spaces.

Example 2.1. *Let $V = \mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$ be the space of n -tuples of real numbers. Scalars are real numbers, addition and subtraction operate component wise, and scalar multiplication operates on each individual entry separately. V is a vector space.*

Definition 2.2. *A vector space L over a field \mathbb{F} is a **Lie algebra** if there is a product*

(called “bracket”) $[\cdot, \cdot] : L \times L \rightarrow L$, such that

1. The bracket is **\mathbb{F} -bilinear**: $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$ for all $x, y, z \in L$ and $a, b \in \mathbb{F}$.
2. The bracket is **alternating**: $[x, x] = 0$ for all $x \in L$.
3. The **Jacobi identity** holds: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

It is important to see that axioms 1 and 2 imply that $[x, y] = -[y, x]$ for all $x, y \in L$ which is called **skew symmetry**. Also, the converse holds if $\text{char}(\mathbb{F}) \neq 2$.

Notice also that the Jacobi identity works as a derivation property. Recall that a derivation is a map d with $d(a \cdot b) = d(a) \cdot b + a \cdot d(b)$. If we consider the adjoint map, ad_a , defined by $ad_a(b) = [a, b]$, we can rewrite the Jacobi identity as:

$$ad_a([b, c]) = [ad_a(b), c] + [b, ad_a(c)],$$

and this may look similar to the product rule for derivatives.

Example 2.3. Let $\mathfrak{gl}_n(\mathbb{F}) = \mathbb{F}^{n \times n}$ be the vector space of all $n \times n$ matrices over a field \mathbb{F} . We can define the bracket by $[A, B] = AB - BA$ for all $A, B \in \mathfrak{gl}_n(\mathbb{F})$. This bracket operation is known as the **commutator bracket** and is the most frequently used bracket operation when working in Lie algebras. Equipped with this bracket $\mathfrak{gl}_n(\mathbb{F})$ becomes a Lie algebra.

Definition 2.4. Given an element x of a Lie algebra L , one defines the **adjoint action** of x on L as the map: $ad_x : L \rightarrow L$ with $ad_x(y) = [x, y]$ for all $y \in L$.

Definition 2.5. A **subalgebra** K of L is a subspace of L which is closed under the Lie bracket. Specifically, for each $x, y \in K$, we must have $[x, y] \in K$ (briefly, $[K, K] \subseteq K$).

Definition 2.6. A Lie algebra L is **abelian** if $[x, x] = 0$ for all $x \in L$ (briefly, $[L, L] = 0$).

Example 2.7. Consider D the collection of all diagonal matrices in $\mathfrak{gl}_n(\mathbb{F})$. Then D is an abelian subalgebra of $\mathfrak{gl}_n(\mathbb{F})$.

Example 2.8. Let $\mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{F} \right\}$. Then $\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = ah + be + cf$,
 where $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\{h, e, f\}$ forms a basis for $\mathfrak{sl}_2(\mathbb{F})$. Using the commutator bracket, we have that:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad \text{and} \quad [e, f] = h.$$

Using this bracket structure $\mathfrak{sl}_2(\mathbb{F}) = \text{span}_{\mathbb{F}}\{h, e, f\}$ is a 3-dimensional Lie algebra.

We call this Lie algebra the **special linear** Lie algebra of 2×2 matrices.

Definition 2.9. Let L be a Lie algebra. An **ideal** I of L , denoted $I \triangleleft L$, is a subspace of L such that for all $x \in I$ and $g \in L$, we have $[g, x] \in I$ (briefly, $[L, I] \subseteq I$).

In Lie algebras, there is no need to differentiate between left and right ideals because of skew symmetry. If we have a left ideal of a Lie algebra L , then for all $x \in I$ and $g \in L$, $[g, x] \in I$. Skew symmetry gives us that $[x, g] = -[g, x] \in I$, so I is also a right ideal. Similarly, right ideals must also be left ideals. Thus all ideals are two-sided in Lie algebras.

Definition 2.10. A non-abelian Lie algebra with no proper non-trivial ideals is called **simple**. In other words, L is simple if $[L, L] \neq 0$ and if I is an ideal of L , then $I = 0$ or $I = L$.

Definition 2.11. Let L be a Lie algebra over field \mathbb{F} . Then $L^{(0)} = L$ and

$$L^{(m)} = [L^{(m-1)}, L^{(m-1)}] \quad \text{for } m = 1, 2, 3, \dots$$

are subalgebras of L with each $L^{(m)}$ an ideal of $L^{(m-1)}$. Additionally, the superset chain

$$L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \dots$$

is called the **derived series** of L . A Lie algebra L is said to be **solvable** if $L^{(m)} = \{0\}$ for some m .

Notice if a Lie algebra L is abelian, then L is solvable. We know this since $L^{(1)} = [L, L] = 0$. We also have if a Lie algebra L is simple then $L = L^{(m)}$ for all m .

We can consider abelian and simple to be at opposite ends of the spectrum: when L is abelian $L^{(k)}$ is always zero (and never L itself), and when L is simple $L^{(k)}$ is always L itself (and never 0).

Definition 2.12. Let L be a Lie algebra over a field \mathbb{F} . The series of ideals

$$L = L^0 \supseteq L^1 \supseteq L^2 \supseteq \dots$$

where $L^{j+1} = [L, L^j]$ for $j = 1, 2, 3, \dots$ is called the **lower central series** of L . A Lie

algebra L is **nilpotent** if $L^m = 0$ for some m , in other words $\underbrace{[[\cdots [[L, L], L], \dots], L]}_{m\text{-times}} = 0$.

Note: From this point forward, we will be working over \mathbb{C} (the complex numbers).

Proposition 2.13. *If a Lie algebra L is abelian then L is nilpotent. If L is nilpotent then L is solvable.*

Proof. If a Lie algebra L is abelian then $[L, L] = L^1 = \{0\}$, so L is nilpotent. Nilpotency implying solvability can be proved using induction. \square

Notice that the converse of Proposition 2.13 does not hold. In other words, solvable does not imply nilpotent, and nilpotent does not imply abelian.

Definition 2.14. *Let $\varphi : L_1 \rightarrow L_2$ be a linear map between two Lie algebras. We call φ a **homomorphism** if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L_1$. We call φ an **isomorphism** if φ is also bijective. We can also define the **kernel** of φ to be the set $\ker(\varphi) = \{x \in L_1 \mid \varphi(x) = 0\}$. The kernel is an ideal of L_1 .*

3 Lie Algebra Modules and the Killing Form

Using our background definitions from section 2, we can now introduce structures called representations and modules. In general, this background is compiled from chapter 8 of [M] and chapter 7 of [EW].

Definition 3.1. *A homomorphism $\varphi : L \rightarrow \mathfrak{gl}(V)$ is called a **representation** of the Lie algebra L on the vector space V . A representation is called **finite dimensional** if the vector space V is finite dimensional.*

Example 3.2. The map $ad : L \rightarrow \mathfrak{gl}(L)$ defined by $ad(x) = ad_x$ for $x \in L$ is a representation of L on itself. This is called the adjoint representation. Additionally, $\ker(ad) = \{x \in L \mid ad(x) = 0\} = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\} = Z(L)$ which is called the center of the lie algebra.

Definition 3.3. A finite dimensional vector space M over \mathbb{C} equipped with a bilinear L -action, say $\bullet : L \times M \rightarrow M$ denoted $(x, \mathbf{v}) \mapsto x \bullet \mathbf{v}$ is an **L-module** if $[x, y] \bullet \mathbf{v} = x \bullet (y \bullet \mathbf{v}) - y \bullet (x \bullet \mathbf{v})$ for all $x, y \in L$ and $\mathbf{v} \in M$. A subspace of a module closed under the action of L is called a **submodule**.

Definition 3.4. Let $\varphi : M_1 \rightarrow M_2$ be a linear map between two L -modules. We call φ an **L-module map** when $\varphi(g \bullet \mathbf{v}) = g \bullet \varphi(\mathbf{v})$ for all $g \in L$ and $\mathbf{v} \in M_1$. Moreover, if φ is bijective then φ is called an (L -module) **isomorphism**.

Lemma 3.5. Let $\varphi : V \rightarrow W$ be an L -module homomorphism. Then

- $\ker(\varphi) = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) = 0\}$ is a submodule of V .
- $\varphi(V) = \{\varphi(\mathbf{v}) \in W \mid \mathbf{v} \in V\}$ is a submodule of W .

Every module leads to a representation and every representation leads to a module.

We can define one from the other by $x \bullet \mathbf{v} = \varphi(x)(\mathbf{v})$ for $x \in L$ a Lie algebra and $\mathbf{v} \in M$ a finite dimensional vector space. Due to this fact, we will use the terms module and representation interchangeably.

Definition 3.6. An **irreducible module** M is a non-trivial module ($M \neq 0$) with no proper non-trivial submodules.

Since the terms *module* and *representation* can be used interchangeably, we can similarly define an *irreducible representation*:

A representation $\varphi : L \rightarrow \mathfrak{gl}(V)$ is **irreducible** if:

1. $V \neq \{0\}$ and
2. $\{0\} \neq U$ a subspace of V and $(\varphi(x))(u) \in U$, for all $u \in U$ and $x \in L$ implies $U = V$.

Example 3.7. Any Lie algebra L is an L -module under the adjoint action. In this case we have $x \bullet y = ad_x(y) = [x, y]$ for all $x, y \in L$. We also have that any submodule of L is an ideal of L from Definition 3.3. Moreover, L is an irreducible L -module if and only if L is simple or one-dimensional.

Lemma 3.8. (Schur's Lemma) Let V be an irreducible L -module and $\psi : V \rightarrow W$ be an L -module homomorphism. Then either $\psi = 0$ or ψ is invertible.

Proposition 3.9. The maximal solvable ideal containing all solvable ideals of L is called the **radical** of L , and it is denoted by $\text{rad}(L)$. A Lie algebra L is **semisimple** if $\text{rad}(L) = \{0\}$.

See Proposition 1.13 in [Ca].

Theorem 3.10. A semisimple Lie algebra is the direct sum of simple Lie algebras.

See Theorem 5.2 in [H1].

If a Lie algebra L is simple, then L is not solvable. We then have $\text{rad}(L) = \{0\}$ (since the radical is a proper ideal and thus trivial) and so L is semisimple. Thus if a Lie algebra L is simple, then L is semisimple.

Definition 3.11. The **Killing form** $\kappa(\cdot, \cdot) : L \times L \rightarrow \mathbb{C}$ is defined by:

$$\kappa(x, y) = \text{Tr}(ad_x \circ ad_y) \text{ for all } x, y \in L.$$

where Tr denotes the trace of a linear transformation.

Theorem 3.12. (Cartan's Criterion I)

Let V be a finite dimensional vector space over a field of characteristic zero and $L \subseteq \mathfrak{gl}(V)$ be a Lie algebra of split linear transformations on V . Then $\kappa(x, y) = \text{Tr}(x \circ y) = 0$ for all $x \in [L, L]$, $y \in L$ if and only if L is solvable.

Corollary 3.13. (Cartan's Criterion II)

Let L be a finite dimensional Lie algebra over an algebraically closed field of characteristic zero. Then: $\kappa(x, y) = \text{Tr}(ad_x \circ ad_y) = 0$ for all $x \in [L, L]$ and $y \in L$ if and only if L is solvable.

See Theorem 6.6 and Corollary 6.7 in [M] for proofs of Cartan's Criterion I and II.

4 Simple Lie Algebras

In this section, we will explore the different types of simple Lie Algebras and how they are classified. Our project focuses on type A_n and D_n , but here we define other types as well, namely, the classical simple Lie algebras.

Material in this section is mainly drawn from chapters 12 and 13 of [EW] and chapters 1 and 8 of [H1]. We refer the reader to explore [EW] for an accessible discussion of these topics, and [H1] for a more advanced approach.

Definition 4.1. $\mathfrak{sl}_n(\mathbb{C}) = \{x \in M_{n+1}(\mathbb{C}) \mid \text{tr}(x) = 0, n \geq 1\}$ is type A_n .

In order to define the classical algebras, we must be equipped with the following definition: Let $J \in gl(\ell, \mathbb{C})$ be a fixed matrix and set

$$gl^J(\ell, \mathbb{C}) = \{ A \in \mathfrak{gl}(\ell, \mathbb{C}) \mid JA + A^T J = 0 \}.$$

Also recall that I_n is the $n \times n$ identity matrix.

Definition 4.2. When $\ell = 2n + 1$ and $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}$, $\mathfrak{gl}^J(\ell, \mathbb{C})$ is denoted by

$\mathfrak{so}(2n + 1, \mathbb{C})$ is called the **odd special orthogonal Lie algebra**. In the classification this is referred to as type **B_n**.

Definition 4.3. When $\ell = 2n$ and $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, the Lie algebra $\mathfrak{gl}^J(\ell, \mathbb{C})$ is denoted by $\mathfrak{sp}(2n, \mathbb{C})$. We classify this type of Lie algebra as type **C_n**, and we call it the **symplectic Lie algebra**.

Definition 4.4. When $\ell = 2n$ and $J = \begin{bmatrix} O & I_n \\ I_n & 0 \end{bmatrix}$, $\mathfrak{gl}^J(\ell, \mathbb{C})$ is denoted by $\mathfrak{so}(2n, \mathbb{C})$. We classify this type of Lie algebra as type **D_n**, and we call it the **even special orthogonal Lie algebra**.

Definition 4.5. Simple Lie algebras of types A_n , B_n , C_n , and D_n are called **classical algebras**, and algebras of types E , F , and G (which we will not define) are called **exceptional algebras**.

Theorem 4.6. (Killing and Cartan's classification of simple Lie algebras).

Each finite dimensional simple Lie algebra (over \mathbb{C}) is isomorphic to one of the following algebras:

A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$), $E_6, E_7, E_8, F_4,$ and G_2 .

For further discussion of the classification, we direct the reader to Theorem 13.1 in [EW].

Alternatively, see Theorem 11.4 in [H1].

Definition 4.7. A **Cartan subalgebra** \mathfrak{h} of a simple Lie algebra \mathfrak{g} is a subalgebra which is nilpotent and self-normalizing (if $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, and $[x, y] \in \mathfrak{h}$ then $x \in \mathfrak{h}$).

*Another way to define a Cartan subalgebra is that it is a maximal toral subalgebra (a **toral** subalgebra is a subalgebra \mathfrak{h} where for all $h \in \mathfrak{h}$, $\text{ad}(h) : \mathfrak{g} \rightarrow \mathfrak{g}$ is diagonalizable). Every Cartan subalgebra of a finite dimensional simple Lie algebra \mathfrak{g} has the same dimension.*

*This dimension is called the **rank** of the simple Lie algebra.*

Proposition 4.8. *The restriction of the Killing form κ to the Cartan subalgebra \mathfrak{h} is nondegenerate.*

See Corollary 8.2 in [H1]

Remark 4.9. *Every toral subalgebra \mathfrak{h} must be abelian. With this fact, we know that for all $x, y \in \mathfrak{h}$, $\text{ad}(x)$ and $\text{ad}(y)$ commute with each other and so the space of endomorphisms $\text{ad}(\mathfrak{h})$ can be simultaneously diagonalized.*

Recall if \mathfrak{h} be a vector space over \mathbb{C} , the **dual space** of \mathfrak{h} is

$$\mathfrak{h}^* = \{f : \mathfrak{h} \rightarrow \mathbb{C} \mid f \text{ is linear}\}.$$

It turns out that we can use the Killing form to induce an inner product on both \mathfrak{h} and \mathfrak{h}^* . From now on we treat both Cartan subalgebras and their duals as Euclidean (i.e. inner product) spaces. We will use (v, w) to denote the inner product of v and w .

Theorem 4.10. *Let \mathfrak{g} be a simple Lie algebra (over \mathbb{C}) and \mathfrak{h} be a Cartan subalgebra.*

Then

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid [h, v] = \alpha(h)v \ \forall h \in \mathfrak{h}\}$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, then α is called a **root** of \mathfrak{g} and \mathfrak{g}_α is its **root space**.

Moreover, $\mathfrak{g}_0 = \mathfrak{h}$ (the 0-space is the Cartan subalgebra).

This is Proposition 12.3 in [EW]. Also, see section 8.1 of [H1] for further discussion of root space decomposition of simple Lie algebras.

We can think of root spaces as simultaneous eigenspaces decomposing \mathfrak{g} and the roots as the eigenvalues corresponding to those eigenspaces. The set of roots has a particularly nice structure which we call a **root system**, usually denoted R .

Proposition 4.11. *Let R be the root system of a Lie algebra \mathfrak{g} relative to a Cartan subalgebra \mathfrak{h} . Define the notation t_α as $t_\alpha \in \mathfrak{h}$ such that $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in \mathfrak{h}$.*

1. R spans \mathfrak{h}^* .

2. If $\alpha \in R$ then $-\alpha \in R$.
3. Let $\alpha \in R$, $x \in \mathfrak{g}$, and $y \in \mathfrak{g}_{-\alpha}$. Then $[x, y] = \kappa(x, y)t_\alpha$
4. If $\alpha \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional, with basis t_α .
5. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$, for $\alpha \in R$.
6. If $\alpha \in R$ and x_α is any nonzero element of \mathfrak{g}_α , then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that x_α , y_α , and $h_\alpha = [x_\alpha, y_\alpha]$ span a three dimensional subalgebra, S_α , isomorphic to $\mathfrak{sl}(2, \mathbb{F})$ via $x_\alpha \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y_\alpha \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $h_\alpha \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
7. $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_{-\alpha})}$; $h_\alpha = -h_{-\alpha}$.

See Proposition 8.3 in [H1].

Proposition 4.12. *Let R be a root system of a Lie algebra \mathfrak{g} relative to a Cartan subalgebra \mathfrak{h} , and Let $S_\alpha \cong \mathfrak{sl}(2, \mathbb{F})$ be a subalgebra of \mathfrak{g} as constructed in part (6) of Proposition 4.11. Then*

1. $\alpha \in R$ implies $\dim(\mathfrak{g}_\alpha) = 1$. In particular $S_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathfrak{h}_\alpha$, ($\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$), and for given nonzero $x_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ satisfying $[x_\alpha, y_\alpha] = h_\alpha$
2. If $\alpha \in R$, the only scalar multiple of α which are roots are α and $-\alpha$.
3. If $\alpha, \beta \in R$, then $\beta(h_\alpha) \in \mathbb{Z}$, and $\beta - \beta(h_\alpha)\alpha \in R$. We call the numbers $\beta(h_\alpha)$

Cartan integers.

4. If $\alpha, \beta, (\alpha + \beta) \in R$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{g}_{\alpha+\beta}$.

5. Let $\alpha, \beta \in R$, $\beta \neq \pm\alpha$. Let r, q , be (respectively) the largest integers for which $\beta - r\alpha, \beta + q\alpha$ are roots. Then all $\beta + i\alpha \in R$ (where $-r \leq i \leq q$), and $\beta(h_\alpha) = r - q$.
6. \mathfrak{g} is generated (as a Lie algebra) by the root spaces \mathfrak{g}_α .

See Proposition 8.4 in [H1].

Theorem 4.13. *Let \mathfrak{g} be a simple Lie algebra with Cartan subalgebra \mathfrak{h} , and root system R . Then:*

1. R spans \mathfrak{h}^* , and 0 does not belong to R .
2. If $\alpha \in R$ then $-\alpha \in R$, but no other scalar multiple of α is a root.
3. If $\alpha, \beta \in R$, then $\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in R$.
4. If $\alpha, \beta \in R$, then $\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

See Theorem 8.5 in [H1].

Note, we will begin using the following notation:

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \langle \beta, \alpha \rangle \tag{4.1}$$

Definition 4.14. *A subset B of a root system R is a **base** for R if:*

1. B is a vector space basis for our inner product space $E = \mathfrak{h}^*$, and
2. every $\beta \in R$ can be written as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}$, where all the non-zero coefficients k_α have the same sign.

We say that a root $\beta \in R$ is **positive with respect to B** if the coefficients given in (2) of Definition 4.14 are positive. Similarly, if all of the coefficients are negative, then β is **negative with respect to B** .

Theorem 4.15. *Every root system has a base.*

This is Theorem 11.10 in [EW].

Definition 4.16. *Let R^+ denote the set of the positive roots in a root system R with respect to a base B , and let R^- denote the set of the negative roots. Then $R = R^+ \cup R^-$, a disjoint union. The set B is contained in R^+ . The elements of B are called **simple roots**.*

Example 4.17. *The smallest simple Lie algebra is type A_1 . This is nothing more than \mathfrak{sl}_2 (the collection of 2×2 matrices with complex entries and trace 0). This example is extremely important to understanding representation theory of finite dimensional Lie algebras because each representation of a simple Lie algebra is built from copies of \mathfrak{sl}_2 . Recall from Example 2.8 our basis elements E, F , and H . We have that $\mathfrak{h} = \text{span}(H)$ is a Cartan subalgebra for \mathfrak{sl}_2 . If we define $\alpha_1 \in \mathfrak{h}^*$ by $\alpha_1(H) = 2$ then α_1 is our simple root and λ_1 is our fundamental weight where $\alpha_1 = 2\lambda_1$.*

Definition 4.18. *Given $\alpha \in \mathfrak{h}^*$ can define*

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha = \beta - \langle \beta, \alpha \rangle \alpha$$

for all $\beta \in \mathfrak{h}^$ where (β, α) is the inner product on \mathfrak{h}^* induced from the inner product on \mathfrak{h} which is in turn derived from the Killing form. The mapping σ_α is a reflection across the hyperplane determined by α .*

The group of invertible linear transformations of the inner product space \mathfrak{h}^* generated by the reflections σ_α for $\alpha \in R$ is known as the **Weyl group** of the root system R .

It is important to note that the Weyl group acts simply transitively on the roots. This means that only the identity fixes the entire root system and there is a Weyl group element which sends any arbitrary root to any other arbitrarily chosen root. This then implies that the Weyl group permutes the roots and is finite. See for example Theorem 1.8 in [H2]. Alternatively, Theorem 4.13 (3) gives us that simple reflections (see Definition 5.2) send roots to other roots, and from this it follows that the Weyl group permutes the set of roots.

5 Representations of Simple Lie Algebras

The material in this section is drawn from chapters 7, 11, and 13 of [H1] and sections 13.1 and chapter 15 of [EW]. More information about the Cartan matrices and Dynkin diagrams can be found in chapter 6 of [Ca].

Definition 5.1. Let $\lambda_1, \dots, \lambda_\ell$ be the dual basis (relative to the inner product on \mathfrak{h}^*):

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}. \text{ We call these } \lambda_i \text{ **fundamental weights** .}$$

Definition 5.2. For each $1 \leq i \leq n$, we call $\sigma_{\alpha_i} = \sigma_i$ a **simple reflection** associated with the simple root α_i .

Note that the Weyl group can be generated by the simple reflections alone. Recalling the definition of the fundamental weights, we have that

$$\sigma_i(\lambda_j) = \lambda_j - \langle \lambda_j, \alpha_i \rangle \alpha_i = \lambda_j - \frac{2(\lambda_j, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i = \lambda_j - \delta_{ij} \alpha_i$$

where δ_{ij} is the **Kronecker delta**, that is $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

Definition 5.3. A **Cartan matrix** of a root system R is the $\ell \times \ell$ matrix with ij -th entry $\langle \alpha_i, \alpha_j \rangle$ (as defined in Equation 4.1). The entries in a Cartan matrix are all integers.

Definition 5.4. The **Dynkin diagram** of a generalised Cartan matrix A is the graph whose vertices are indexed by the row of the matrix and whose edges are described as follows:

- If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ lines equipped with an arrow pointing toward i if $|a_{ij}| > 1$;
- If $a_{ij}a_{ji} > 4$, the vertices i and j are connected by a line colored by the ordered pair of integers $|a_{ij}|, |a_{ji}|$.

Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . The simple roots $B = \{\alpha_1, \dots, \alpha_n\}$ form a basis for \mathfrak{h}^* . The fundamental weights $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ form another basis for \mathfrak{h}^* . The root and weight bases are related by the Cartan matrix. Specifically, if $C = (c_{ij})$, then

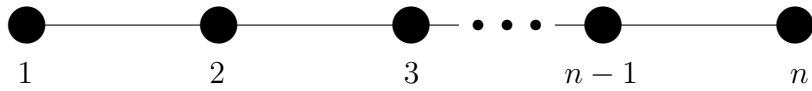
$$\alpha_j = \sum_{i=1}^{\ell} c_{ij} \lambda_i.$$

Let us list the Cartan matrices and Dynkin diagrams of the classical simple Lie algebras: A_n, B_n, C_n , and D_n .

Type A_n algebras have Cartan matrices of the form:

$$A_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

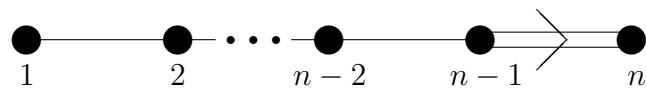
The Dynkin diagrams of type A_n have the form:



Type B_n algebras have Cartan matrices of the form:

$$B_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -2 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

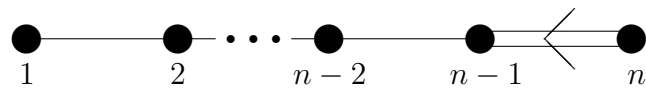
The Dynkin diagrams of type B_n have the form:



Type C_n algebras have Cartan matrices of the form:

$$C_n = B_n^T = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & 0 & -2 & -2 \end{bmatrix}$$

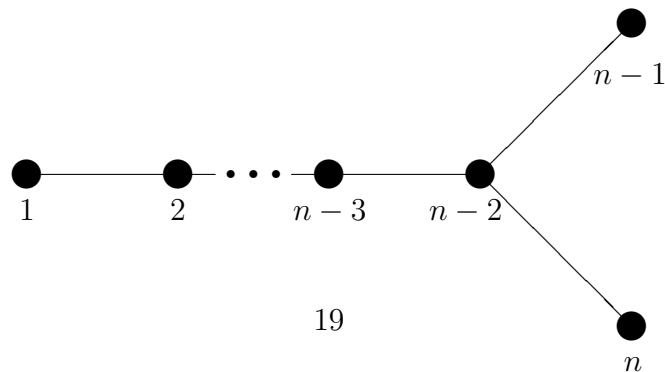
The Dynkin diagrams of type C_n have the form:



Type D_n algebras have Cartan matrices of the form:

$$D_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 2 & -1 & 0 & 0 \\ 0 & \ddots & -1 & 2 & -1 & -1 \\ \vdots & \ddots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & -2 \end{bmatrix}$$

The Dynkin diagrams of type D_n have the form:



Theorem 5.5. *Let \mathfrak{g} be a finite dimensional simple Lie Algebra over \mathbb{C} , and let V be a finite dimensional \mathfrak{g} -module. Then,*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

where

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v \quad \forall h \in \mathfrak{h}\}$$

If $V_\lambda \neq \{0\}$, then λ is called a **weight** of V and V_λ is its **weight space**.

See section 7.1 of [H1] and section 15.1.1 of [EW] for further discussion of the weight space decomposition.

Just like we are able to think of roots and root spaces as simultaneous eigenspaces and their corresponding eigenvalues, we can also compare weights to eigenvalues, and weight spaces to eigenspaces. As with roots, we also have that the Weyl group permutes the set of weights. In other words, the Weyl group sends weights of a representation to other weights of that representation.

Definition 5.6. *For any \mathfrak{g} -module M , we know that M decomposes into weight spaces: M_λ for $\lambda \in \mathfrak{h}^*$. The dimension of a weight space M_λ is called the **multiplicity** of the weight λ .*

Theorem 5.7. *Let M be an irreducible \mathfrak{g} -module. There exists a (unique) weight $\lambda \in \mathfrak{h}^*$ of M such that given any other weight $\mu \in \mathfrak{h}^*$ we have $\mu = \lambda - \sum_{i=1}^n b_i \alpha_i$ where the b_i 's are non-negative integers. In other words, every other weight can be obtained by subtracting collections of positive roots from this weight λ called the **highest weight** of M .*

See Theorem 10.20 in [Ca], or alternatively, see Lemma 15.3 of [EW].

Theorem 5.8. *Any two irreducible modules with the same highest weight are isomorphic. The converse also holds: Any two isomorphic irreducible modules have the same highest weight.*

See Theorem 10.21 of [Ca]. chapter 10 in [Ca] provides more information on irreducible modules.

Definition 5.9. *Let $L(\lambda)$ be a finite dimensional irreducible \mathfrak{g} -module with highest weight $\lambda \in \mathfrak{h}^*$ and $\lambda \neq 0$. Then λ is a **minuscule weight** and $L(\lambda)$ is a **minuscule module** if the Weyl group $W(\mathfrak{g})$ acts transitively on the set of weights of $L(\lambda)$.*

If $\mu = w(\lambda)$ for $\mu, \lambda \in \mathfrak{h}^*$ and $w \in W$, then the corresponding weight spaces M_μ and M_λ have the same dimension. It follows that weights in the Weyl orbit all have the same multiplicity. Notice also that in an irreducible module, the highest weight is always one-dimensional, and so the dimension of a minuscule module is the same as the number of its weights.

We have from [H1] a very useful theorem regarding the minuscule weights of the finite dimensional simple Lie algebras.

Theorem 5.10. Minuscule weights for finite dimensional simple Lie algebras

<i>Type:</i>	A_n	B_n	C_n	D_n	E_6	E_7
<i>Minuscule Weights:</i>	$\lambda_1, \dots, \lambda_n$	λ_n	λ_1	$\lambda_1, \lambda_{n-1}, \lambda_n$	λ_1, λ_6	λ_7

Type F_4 , E_8 , and G_2 algebras have no minuscule weights.

This theorem is provided in the form of a table in Exercise 13 of section 13.4 of [H1].

6 Seeing Irreducibility Through Cycle Structures

Now that we have the necessary background, we can delve into our project objective. The question we would like to answer is: can we see the irreducibility of a minuscule module through the cycle structures of its Weyl group alone? This question has been previously investigated for some of the minuscule weights of various finite dimensional simple Lie Algebras.

In a different project, [CMS] investigated the cases of the first minuscule weight, λ_1 , of types A_n ($n \geq 1$), C_n ($n \geq 3$), D_n ($n \geq 4$), as well as E_6 and E_7 . Their results are provided in the following paragraphs.

If we look at the set of weights for $L(\lambda_1)$ for type A_n , then the set of weights is of size $(n + 1)$. We see because the Coxeter element is an $(n + 1)$ -cycle, the only allowed dimensions for submodules are 0 and $(n + 1)$. Therefore, this minuscule representation must be irreducible. This follows from cycle structure alone! This is also the case with $L(\lambda_n)$ because of the symmetry in the Dynkin Diagram.

Type C_n has only one minuscule weight: λ_1 . The set of weights for $L(\lambda_1)$ for type C_n is of size $2n$. In this project, [CMS] found that the Coxeter element is a $2n$ -cycle, allowing for submodules of dimension 0 or $2n$, thus showing the irreducibility of $L(\lambda_1)$ using cycle structures alone.

Investigating the first minuscule weight (λ_1) of type D_n , we have that the set of weights is, similar to type C_n , of size $2n$. The authors of [CMS] found that there will be one element of a pair of n -cycles, and a second element of one $(2n - 2)$ -cycle and a 2-cycle. This would only allow for submodules of dimension 0 or $2n$, showing the irreducibility of

$L(\lambda_1)$ through cycle structures alone.

The only exceptional algebras that have minuscule weights are types E_6 and E_7 . Type E_6 has two minuscule weights, λ_1 and λ_6 . For E_6 and λ_1 we have the dimension of $L(\lambda_1)$ is 27. The authors of [CMS] showed that this minuscule module is irreducible using the occurrence of an element with 2×12 -cycles and a 3-cycle with a second element made of 3×9 -cycles. Intersecting the possible submodule dimensions that these two elements would allow, we have that the only possibilities are 0 and 27. This result can be extended to $L(\lambda_6)$ due to the symmetry in the Dynkin diagram of type E_6 . Looking at type E_7 , there is only one minuscule weight: λ_7 . This set of weights is of size 42. We can see the irreducibility of $L(\lambda_7)$ using two elements: one element made of 3×18 -cycles and a 3-cycle, and another made of 3×14 -cycles. These two elements show that the only possible submodule dimensions are 0 and 42, showing the irreducibility of $L(\lambda_7)$ using only cycle structures.

After the publication of [CMS], the only remaining minuscule weights to investigate were $\lambda_2, \dots, \lambda_{n-1}$ of type A_n , λ_{n-1} and λ_n of type D_n , and λ_n of type B_n . The paper [CH] explored showing irreducibility of type B_n 's only minuscule weight λ_n and found that for many ranks of type B_n it is impossible to show irreducibility using cycle structures alone. In [CH], the authors successfully showed that we can see irreducibility through cycle structures alone for $n = 2, 3, 5$, and 7. The approach fails for $n = 4, 6, 8, 9, 10, 11$, and 12. This paper, [CH], conjectures that the approach continues to fail for higher ranks.

Our project specifically looks into the remaining cases for types A_n and D_n , as mentioned above. From this point forward, we will use the notation $W(\lambda_i)$ to indicate the

set of weights in the orbit of λ_i , and $|W(\lambda_i)|$ its dimension.

All fundamental weights of type A_n are minuscule weights, so we must examine the cycle structures of permutations on $W(\lambda_1), \dots, W(\lambda_n)$. Note that in type A_n the Dynkin diagram is symmetric, so action on $W(\lambda_i)$ and $W(\lambda_{n-i})$ is essentially the same. We only need to look at half of the weights to make claims about them all.

When investigating type D_n we have three minuscule weights to investigate: λ_1, λ_{n-1} and λ_n . There is also symmetry in D_n 's Dynkin diagram, so actions on $W(\lambda_{n-1})$ and $W(\lambda_n)$ are essentially the same.

Our project produced the results found in the following table. The notation c_i^j represents a j cycles of length i . For example in type A_2 the minuscule weight λ_1 can be shown is irreducible by c_3 , or one cycle of length 3.

Irreducibility of type A_n Classical Algebra				
Algebra Type	Minuscule Weight (λ_k)	$ W(\lambda_k) $	Cycle Structure of Coxeter Element	Cycle Structures Used to Show Irreducibility
A_2	λ_1	3	c_3	c_3
A_3	λ_1	4	c_4	c_4
	λ_2	6	c_4c_2	c_4c_2 and c_3^2
A_4	λ_1	5	c_5	c_5
	λ_2	10	c_5^2	c_5^2 and $c_4^2c_2$
A_5	λ_1	6	c_6	c_6
	λ_2	15	$c_6^2c_3$	$c_6^2c_3$ and c_5^3
	λ_3	20	$c_6^3c_2$	$c_6^3c_2$ and c_5^4
A_6	λ_1	7	c_7	c_7
	λ_2	21	c_7^3	c_7^3 and $c_3c_6^3$
	λ_3	35	c_7^5	c_7^5 and c_5^7
A_7	λ_1	8	c_8	c_8
	λ_2	28	$c_8^3c_4$	$c_8^3c_4$ and c_7^4
	λ_3	56	c_8^7	c_8^7 and c_7^8
	λ_4	70	$c_8^8c_4$	$c_8^8c_4$ and c_7^{10}
A_8	λ_1	9	c_9	c_9
	λ_2	36	c_9^4	c_9^4 and $c_8^4c_4$
	λ_3	84	$c_9^9c_3$	$c_9^9c_3$, c_7^{12} , and $c_8^{10}c_4$
	λ_4	126	c_9^{14}	c_9^{14} , c_5^{14} , and $c_8^{15}c_4c_2$
A_9	λ_1	10	c_{10}	c_{10}
	λ_2	45	$c_{10}^4c_5$	$c_{10}^4c_5$ and c_9^5
	λ_3	120	c_{10}^{12}	c_{10}^{12} , $c_8^{14}c_4^2$, and $c_9^{13}c_3$

Irreducibility of type D_n Classical Algebra				
Algebra Type	Minuscule Weight (λ_k)	$ W(\lambda_k) $	Cycle Structure of Coxeter Element	Cycle Structures Used to Show Irreducibility
D_4	λ_1	8	c_6c_2	c_6c_2 and c_4^2
D_5	λ_1	10	c_8c_2	c_8c_2 and c_6c_4
	λ_5	16	c_8^2	c_8^2 and $c_{12}c_4$
D_6	λ_1	12	$c_{10}c_2$	$c_{10}c_2$ and c_8c_4
	λ_6	32	$c_{10}^3c_2$	$c_{10}^3c_2$ and c_8^4
D_7	λ_1	14	$c_{12}c_2^8$	$c_{12}c_2$ and $c_{10}c_4$
	λ_7	64	$c_{12}^5c_4$	$c_{20}^3c_4$ and $c_7^9c_1$

From the results in [CMS], we know that for type A_n , λ_1 (and thus λ_n) only requires one element of the Weyl group to show irreducibility, but all other weights will require at least two elements to show irreducibility. For these other weights, we must examine

them case-by-case. We conclude with several examples:

Example 6.1. A_3, λ_2 : *This type and weight requires two different elements to show irreducibility. Since $|W(\lambda_2)| = 6$ we need to show that the only possible submodule dimensions are 0 and 6.*

- *The Coxeter element has 1×4 -cycle and 1×2 -cycle. This allows for submodules of size 0, 2, 4, and 6.*
- *Another element is made up of 2×3 -cycles, and that allows for submodules of size 0, 3, and 6.*

When we intersect the submodule dimensions that these two elements allow, we only have dimension 0 and 6, so this minuscule module $L(\lambda_2)$ is irreducible. There is no equivalent weight by symmetry of the Dynkin diagram.

Example 6.2. A_8, λ_4 : *This set of weights has $|W(\lambda_4)| = 126$.*

- *The Coxeter element has structure 14×9 -cycles. This allows for submodules of dimension 0, 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, and 126.*
- *We also have an element structured by 5×14 -cycles and 8×7 -cycles. This allows for submodule dimensions in multiples of seven: 0, 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105, 112, 119, 126.*

Intersecting these possibilities, we are left with possible submodule dimensions 0, 63, and 126.

- *We have another element with structure 15×8 -cycles, 1×4 -cycles, 1×2 -cycles,*

this element will only allow for even submodule dimension, ruling out 63 as a possibility.

This will leave us with possible submodule dimension 0 and 126, showing irreducibility. In A_8 , λ_4 is equivalent to λ_5 by symmetry. These weights require 3 elements to show irreducibility using cycle structure.

Example 6.3. D_5, λ_1 : *In this case we have that $|W(\lambda_1)| = 10$ so we need to show that the only possible submodule dimensions are 0 and 10.*

- *The Coxeter element has structure 1×8 -cycle and 1×2 -cycle, allowing submodule dimension 0, 2, 8 and 10.*
- *We have another element with cycle structure 1×6 -cycle and 1×4 -cycle so this allows for submodule sizes 0, 4, 6, and 10.*

Intersecting these possibilities we only have possible submodule dimensions 0 and 10, so this minuscule module $L(\lambda_1)$ is irreducible.

Example 6.4. D_5, λ_4 : *In this case we have that $|W(\lambda_1)| = 16$ so we need to show that the only possible submodule dimensions are 0 and 16.*

- *The Coxeter element has structure 2×8 -cycles, allowing submodule dimension 0, 8 and 16.*
- *We have another element with cycle structure 1×12 -cycle and 1×4 -cycle so this allows for submodule sizes 0, 4, 12, and 16.*

Intersecting these possibilities we only have possible submodule dimensions 0 and

16, so this weight is irreducible. Under the symmetry of the Dynkin diagram, we can also say that $L(\lambda_5)$ is also irreducible.

Finally, we can conclude from these results that irreducibility of the minuscule representations of type A_n can be seen from cycle structures alone when $n = 1, 2, 3, 4, 5, 6, 7, 8$, and part of 9. We could not continue past $n = 9$ with our computer code [Co]. We conjecture that one can successfully show irreducibility of type A_n for all n using cycle structure alone, but the number of elements required to show irreducibility will increase. We also expect that this problem will grow more difficult as one approaches the weights toward the middle of the Dynkin diagram.

In type D_n we can see irreducibility of the minuscule representation from cycle structures alone when $n = 4, 5, 6$, and 7. We could not continue past $n = 7$ with our current computer code. We conjecture that you can successfully show irreducibility of type D_n for all n using cycle structure alone.

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