Part 1: 25 multiple-choice questions - 1 hour and 10 minutes. Each question is:

- 4 points for a correct answer,
- 1 point for a blank answer, and
- 0 points for an incorrect answer.

Color in your answers on the scantron sheet.

Part 2: 3 short-answer questions - 15 minutes. Each question is worth a possible 8 points. Write your answers on the answer sheet.

Ancient Chinese Mathematics

The above figure is taken from the Chinese text *Seven Chapters on the Mathematical Art*. This handbook has "... 246 problems intended to provide methods to be used to solve everyday problems of engineering, surveying, trade, and taxation" and is believed to have originated around 200 B.C. The figure shows a square being broken up into 4 similar triangles and one square. Can you see how they could argue the sides of the large square were 5 units, and thus, knew 3-4-5 formed a Pythagorean Triple?

See http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Nine_chapters.html for more information.
PART 1 - MULTIPLE CHOICE QUESTIONS

1. What is the sum of the positive integer divisors of 48?
   (A) 75  (B) 110  (C) 123  (D) 124  (E) None of these

Solution: The positive divisors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48. These sum to 124.

2. What is the last digit of the number \(2008^{2008}\)?
   (A) 2  (B) 4  (C) 6  (D) 8  (E) None of these

Solution: The last digits in the powers of 8 cycle between the digits 8, 4, 2, 6. Since 2008 \(\equiv 0\) \(\mod 4\), we have \(2008^{2008}\) \(\equiv 8^{2008}\) \(\mod 10\). Since 2008 \(\equiv 0\) \(\mod 4\), 2008 \(\equiv 0\) \(\mod 8\), both 2008 \(\equiv 0\) \(\mod 4\) and \(2008 \equiv 0\) \(\mod 8\) are satisfied. Thus 2008 \(\equiv 0\) \(\mod 8\), and 2008 \(\equiv 0\) \(\mod 8\) is a complete cycle so ends in an 6, and also ends in an 6.

3. Let \(L\) denote the line perpendicular to the line \(y = \frac{8}{5}x - 4\) that passes through the point \((12,-6)\). What is the \(x\)-intercept of \(L\)?
   (A) \(-\frac{100}{9}\)  (B) \(\frac{12}{7}\)  (C) 2  (D) \(\frac{17}{9}\)  (E) None of these

Solution: The perpendicular line has the form \(y = -\frac{5}{8}x + b\) and plugging in the point \((12,-6)\) we get \(-6 = -\frac{5}{8}(12) + b\) or \(b = \frac{3}{2}\). To find the \(x\)-intercept, we need to solve the equation \(0 = -\frac{5}{8}x + \frac{3}{2}\) or \(x = \frac{12}{5}\).

4. For all values of \(x\) and \(a\) where it is defined, what is the expression equivalent to?
   \((\sqrt{a - x} + \sqrt{a + x})(\sqrt{a - x} - \sqrt{a + x})\)

(A) \(-2x\)  (B) \(2a\)  (C) \(2x + 2a\)  (D) 0  (E) None of these

Solution: Multiplying this out gives \((a - x) - (a + x) = -2x\).

5. There are 9000 natural numbers with 4 digits in them. How many of them have at least one 5?
   (A) 2765  (B) 2882  (C) 3148  (D) 4000  (E) None of these

Solution: First, count the numbers with no 5 in them. There can be 8 digits in the first place (no 0 or 5) and 9 possible digits in each of the other 3 places. So there are \(8 \cdot 9 \cdot 9 \cdot 9 = 5832\) numbers with no 5s or 9000 – 5832 = 3168 numbers with at least one 5.

6. Let \(f(x) = \frac{x - 3}{x - 2}\). Define a sequence \(f_1(x) = f(x), f_2(x) = f \circ f(x), f_3(x) = f \circ f \circ f(x), \) etc... What is \(f_{100}(x)\) for values of \(x\) where \(f(x)\) is defined?
   (A) \(x\)  (B) \(\frac{2x - 3}{x - 1}\)  (C) \(\frac{x - 3}{x - 2}\)  (D) \(\frac{x - 300}{x - 200}\)  (E) None of these

Solution: You can see \(f_2(x) = f \circ f(x) = \frac{\left(\frac{x - 3}{x - 2}\right) - 3}{\left(\frac{x - 3}{x - 2}\right) - 2} = \frac{2x - 3}{x - 1}\). Then \(f_3(x) = \left(\frac{2x - 3}{x - 1}\right) - 3 = x\). Therefore, \(f_3(x) = f_1(x)\), so the composition makes a cycle through the three functions. Since 100 has a remainder of 1 when divided by 3, \(f_{100}(x) = f_1(x) = f(x)\).

7. Suppose each of the 11 letters in the word APPALACHIAN are placed on a tile and each of the 11 tiles are placed in a bag. If you randomly choose 4 tiles from the bag, one at a time, and line them from left to right as they are chosen, what is the probability you would spell the word PAPA?
   (A) \(\frac{1}{330}\)  (B) \(\frac{1}{495}\)  (C) \(\frac{1}{792}\)  (D) \(\frac{1}{990}\)  (E) \(\frac{1}{1320}\)

Solution: There is a \(\frac{4}{11}\) chance of getting a letter P with the first choice, followed by a \(\frac{3}{10}\) chance of getting an A, then a \(\frac{2}{9}\) chance of getting a letter P with the third choice, followed by a \(\frac{1}{8}\) chance of getting an A with the fourth choice. Multiplying these together and canceling factors gives \(\frac{1}{330}\).
8. What is the product of the solutions of the equation $4(2^x) - 17(2^x) + 4 = 0$?
   (A) 1   (B) $-5/2$   (C) $-4$   (D) $-1$   (E) None of these

Solution: Substituting $y = 2^x$, our equation becomes $4y^2 - 17y + 4 = 0$ which can be factored as $(4y-1)(y-4) = 0$ or $y = \frac{1}{4}$ and $y = 4$. So the two solutions are $2^x = \frac{1}{4}$ or $x = -2$ and $2^x = 4$ or $x = 2$. The product is $-4$.

9. Below is a graph of two functions, labeled $f(x)$ and $g(x)$. Assuming that all of the expressions below are integers, which have a value equal to 1?

(I) $5f(0) - g(0)$
(II) $f \circ g^{-1}(2)$
(III) $|f \circ g^{-1}(8)|$
(IV) $(f + g)(3) - (f - g)(3) + 1$

(A) Only I   (B) Only I and II   (C) Only I and IV
(D) Only II and III   (E) I, II, III, and IV

Solution: $f(0) = 1$ and $g(0) = 4$ so I is true. $f \circ g^{-1}(2) = f(1) = 2$ so II is false. $|f \circ g^{-1}(8)| = |f(-2)| = 5$ so III is false and $(f + g)(3) - (f - g)(3) + 1 = 2g(3) + 1 = -3$ so IV is false.

10. If $a$ and $b$ are positive integers, we write $a|b$ to mean that $a$ is a factor of $b$. Which of the following statements are true for all positive integers $a, b, c,$ and $d$?

(I) $a|b$ and $b|c$ implies $a|c$
(II) $a|b$ and $b|c$ implies $a|(bc)$
(III) $a|b$ and $a|c$ implies $b|c$
(IV) $a|c$ and $b|c$ implies $(ab)|c$
(V) $a|b$ implies $(ac)|(bc)$
(VI) $a|b$ implies $b|a$

(A) Only I and II   (B) Only I and IV, and V   (C) Only I and IV
(D) Only I, II and III   (E) I, II, III, IV, V, and VI

Solution: I, II and V are true. III is not true since $3|21$ and $3|15$ but $21$ isn’t a factor of $15$. IV is not true since $4|12$ and $6|12$ but $24$ isn’t a factor of $12$. VI isn’t true because $3$ is a factor of $6$, but $6$ isn’t a factor of $3$.

11. Which of the following does NOT reduce to $\cos x$ for every $x$ where the expression is defined?

(A) $\frac{\cot x}{\csc x}$   (B) $\frac{\cos x}{\csc^2 x - \cot^2 x}$   (C) $\sec x - \tan x \sin x$
(D) $\frac{\cos^2 x \csc x}{\cot x}$

(E) all of the above reduce to $\cos x$

Solution: A is a simple reduction and B is true since $\csc^2 x = \cot^2 x + 1$. C is true because $\sec x - \tan x \sin x = \frac{1 - \sin^2 x}{\cos x}$ and D is a simple substitution. So they are all true.
12. Assume \( a \) is a number such that the equation \( \sqrt{x-2} - \sqrt{x-3} + a = 0 \) has a solution. What is the solution in terms of \( a \)?

\[
\begin{align*}
(A) & \quad \frac{a^2 + 11}{4} \quad \text{(B)} & \quad \frac{1}{4} a^2 + 10a + 1 \\
(C) & \quad \frac{3a^4 + 6a + 3}{6a^2} \quad \text{(D)} & \quad \frac{1}{4} a^4 + 10a^2 + 1 \\
(E) & \quad \frac{a^4 + 10a^2 + 1}{4a^2}
\end{align*}
\]

**Solution:** \( \sqrt{x-2} = \sqrt{x-3} - a \) so squaring both sides gives \( x - 2 = x - 3 - 2a\sqrt{x-3} + a^2 \) which reduces to \( 1 - a^2 = -2a\sqrt{x-3} \). Squaring both sides again gives \( 1 - 2a^2 + a^4 = 4a^2(x - 3) \). Solving for \( x \) gives answer E.

13. Let \( X \) be a matrix such that \( \left( \begin{array}{cc} -1 & 3 \\ 2 & -1 \end{array} \right) \) and \( g \) be a matrix such that \( \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right) \). What is \( X\)?

\[
\begin{align*}
(A) & \quad \left( \begin{array}{cc} 4 & 17 \\ 2 & -3 \end{array} \right) \quad \text{(B)} & \quad \left( \begin{array}{cc} -3 & 11 \\ -2 & 3 \end{array} \right) \\
(C) & \quad \left( \begin{array}{cc} -3 & 17 \\ 2 & -3 \end{array} \right) \quad \text{(D)} & \quad \left( \begin{array}{cc} 4 & 11 \\ -2 & 3 \end{array} \right) \\
(E) & \quad \text{None of these}
\end{align*}
\]

**Solution:** The product of the matrices on the right is \( \left( \begin{array}{cc} 1 & 28 \\ 0 & 1 \end{array} \right) \). The square of the matrix on the left is \( \left( \begin{array}{cc} 7 & -6 \\ -4 & 7 \end{array} \right) \). Solving the matrix equation gives answer C.

14. If \( f(x) = \ln(15 - x) \) and \( g(x) = |x^2 - 9| \), then what is the domain of \( (f \circ g)(x) \)?

\[
\begin{align*}
(A) & \quad x < -2\sqrt{6} \text{ and } x > 2\sqrt{6} \quad \text{(B)} & \quad -2\sqrt{6} < x < 2\sqrt{6} \\
(D) & \quad x < -5 \text{ and } x > 5 \\
(E) & \quad \text{None of these}
\end{align*}
\]

**Solution:** The domain of \( f(x) \) is \( x < 15 \), so we need to solve \( g(x) = |x^2 - 9| < 15 \). For \( x \). This means \( x^2 < 9 < 15 \) and \( x^2 - 9 > -15 \). The first inequality is true if \( x^2 < 24 \) or \( x > -6 \). The second inequality is true if \( x > -6 \) which holds for all values \( x \).

15. If \( A = (1031)_4 \) and \( B = (221)_4 \) are two numbers in base 4, what is their product in base 3?

\[
\begin{align*}
(A) & \quad (11022221)_3 \quad \text{(B)} & \quad (10221201)_3 \\
(C) & \quad (11101121)_3 \quad \text{(D)} & \quad (11110021)_3 \\
(E) & \quad \text{None of these}
\end{align*}
\]

**Solution:** \( A = 1 \cdot 4^3 + 0 \cdot 4^2 + 2 \cdot 4 + 1 = 77 \) and \( B = 2 \cdot 4^2 + 2 \cdot 4 + 1 = 41 \). Therefore, \( AB = 77 \cdot 41 = 3157 = 11022221 \) in base 3.

16. What is the inverse of the function \( f(x) = \frac{1}{2} \ln(x - \sqrt{x^2 - 1}) \)?

\[
\begin{align*}
(A) & \quad \frac{1}{2} (e^{2x} + e^{-2x}) \quad \text{(B)} & \quad 2 (e^{2x} + e^{-2x}) \\
(C) & \quad \frac{1}{2} (e^{x} - e^{-x}) \quad \text{(D)} & \quad 2 (e^{x} + e^{-x}) \\
(E) & \quad \frac{1}{2} (e^{2x} - e^{-2x})
\end{align*}
\]

**Solution:** Solving the equation \( x = \frac{1}{2} \ln(y - \sqrt{y^2 - 1}) \) for \( y \), we see \( e^{2x} = y - \sqrt{y^2 - 1} \) so \( y - e^{2x} = \sqrt{y^2 - 1} \) or squaring both sides, we get \( y^2 - 2ye^{2x} + e^{4x} = y^2 - 1 \). Canceling the \( y^2 \) terms and solving for \( y \) gives \( 1 + e^{4x} = 2ye^{2x} \) or \( e^{2x} + e^{-2x} = 2y \) or (A).
17. In the picture below, the segments $AB$ and $CD$ are parallel. The measures of the angles in degrees are given in the picture. What is the value of $x$?

Solution: Extend segment $EF$ down to the segment $CD$ to get point $H$. Then $\angle(FEA) = 4x - 9$. Also, $\angle(HFG) = 180 - 5x$ and $\angle(FGH) = 180 - (2x + 3)$. Therefore, triangle $FHG$ has sides $180 - 5x$, $4x - 9$, and $180 - (2x + 3)$. Since they add to 180, this gives $-3x - 12 = -180$ or $x = 56$.

18. Two telephone poles, $x$ and $y$ feet tall are placed $M$ feet apart. Lines are drawn from the top of each pole to the bottom of the other pole. How many feet (along the ground) from the pole of height $x$ will the lines from the two poles intersect?

Solution: Look at the diagram below:

By similar triangles, we have $\frac{x}{y} = \frac{M}{A}$ so $M = \frac{Ax}{H}$. Also, $\frac{y}{B} = \frac{M}{B} = \frac{By}{x}$. Thus, we see $yB = Ax = (M - B)x$ or $B(x + y) = Mx$ and answer B follows.

19. The constants $a$ and $b$ are both between 0 and 1 and satisfy the following:

\[ \sum_{n=0}^{\infty} a^b = \frac{3}{20} \] and \[ ab + b = \frac{9}{48} \]

What is the value of $a + b$ in lowest terms?

Solution: By the geometric series, \[ \sum_{n=0}^{\infty} a^b = \frac{a}{1 - b} = \frac{3}{20} \] The second equation in the problem says that $a = \frac{9}{48} - 1$ and substituting into the geometric equation gives \[ \frac{9}{48} - 1 - \frac{1}{1 - b} = \frac{3}{20} \text{ or } \frac{3 - 16b}{16b(1 - b)} = \frac{3}{20} \] The resulting quadratic equation is $12b^2 - 92b + 15 = 0$ or $(6b - 1)(2b - 15) = 0$ or $b = \frac{1}{6}$ and $b = \frac{15}{2}$. These result in $a = \frac{1}{8}$ and $a = -\frac{39}{40}$ the last is impossible since $a > 0$. Therefore, the only solution is $a = \frac{1}{8}$ and $b = \frac{1}{6}$ which sum to $\frac{14}{48} = \frac{7}{24}$ in lowest terms.

20. The lengths of the sides of a triangle are the roots of $x^3 - 12x^2 + 47x - 60$ and they are all natural numbers. What is the area of the triangle?
Solution: The polynomial can be factored into \((x - 3)(x - 4)(x - 5)\) so the sides of the triangle are 3, 4, and 5. Therefore, it is a right triangle with base 3 and height 4 which will have area 6.

21. A store has a going out of business sale. During the first week, they reduce the list prices by 40%. Then for each of the next 3 weeks, they take additional 10% off the current price. You walk into the store during the fourth week and notice a table for $79.95. What was its price before the sale rounded to the nearest dollar?

(A) $165.00  
(B) $183.00  
(C) $201.00  
(D) $219.00  
(E) $267.00

Solution: The initial price = \(((\text{final price}/0.6)/0.9)/0.9)/0.9\). Therefore, we get \(((79.95/0.6)/0.9)/0.9) = 182.83.

22. Let \((x - 1), (x^2 + x - 2), (x^2 - 1), (x^2 + 3x + 2), \) and \((x^4 - 1)\) be factors of the polynomial \(p(x)\). What is the smallest possible degree of \(p(x)\)?

(A) 5  
(B) 7  
(C) 8  
(D) 9  
(E) 11

Solution: The polynomials can be factored as follows: \((x^2 + x - 2) = (x + 1)(x - 1), (x^2 - 1) = (x + 1)(x - 1), (x^2 + 3x + 2) = (x + 1)(x + 2), \) and \((x^4 - 1) = (x^2 + 1)(x - 1)(x + 1)\). The unique factors are \((x^2 + 1)\), \((x - 1)\), \((x + 1)\), \((x + 2)\) so the smallest degree is the sum of the largest exponents which is 5.

23. Simplify \(\frac{x(x + 1) - ax - a}{(x + 1)(x - a)}\) completely for all values where it is defined.

(A) \(\frac{1 - a}{x - a}\)  
(B) \(x - a\)  
(C) \(\frac{x + 2a}{x - a}\)  
(D) 1  
(E) None of these

Solution: \(\frac{x(x + 1) - ax - a}{(x + 1)(x - a)} = \frac{x(x + 1) - a(x + 1)}{(x + 1)(x - a)} = \frac{(x - a)(x + 1)}{(x + 1)(x - a)} = 1\)

24. If \(9^y = 27\), then what is \(3^{-y}\)?

(A) \(\frac{1}{3}\)  
(B) \(-\frac{1}{3}\)  
(C) \(\frac{1}{3\sqrt{3}}\)  
(D) \(-\frac{1}{3\sqrt{3}}\)  
(E) None of these

Solution: \(9^y = 3^6 = 3^3\) so \(3^y = 3^{1/2}\). Therefore, \(3^{-y} = 3^{-1/2} = \frac{1}{\sqrt{3}}\) which is none of the above.

25. In the figure below, \(BC\) is tangent to the circle at point \(B\). If the length of \(AB\) is \(2\sqrt{3}\) and the measure of angle \(ACB\) is 30 degrees, what is the area of triangle \(ABC\) that lies outside the circle?

(A) \(12\sqrt{3} - \pi\)  
(B) \(6\sqrt{3} - 2\pi\)  
(C) \(6\sqrt{3} - \pi\)  
(D) \(2\pi - \sqrt{3}\)  
(E) None of these

Solution: If the radius of the circle is \(2\sqrt{3}\) and \(\angle BAC = \frac{\pi}{6}\), then the area of the sector \(BAD\) is \(\frac{1}{6}\) of the area of the circle which is \(\frac{1}{6}\pi(2\sqrt{3})^2 = 2\pi\). We have \(\tan(30) = \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{BC}\) so \(BC = 6\). Thus the area of triangle \(ABC\) is \(\frac{1}{2}(6)(2\sqrt{3}) = 6\sqrt{3}\). Therefore, the area of the triangle outside the circle = area of triangle - area inside circle = \(6\sqrt{3} - 2\pi\).
Part 2

1. Barkley the dog is tied halfway between 2 corners of a 12 foot long wall with a 16 foot rope as shown in the picture below.

If $\angle BCD = 90^\circ$, $\angle ABC = 135^\circ$, and walls $AB$ and $CD$ are 12 and 16 feet respectively, how much total area does Barkley have to run around in? Show your steps.

**Solution:** Barkley makes a semicircle of radius 16 on edge BC which has area $\frac{1}{2} \pi 16^2 = 128\pi$. If he turns corner C, he can run around in an area that is one quarter of a circle of radius 10 which is $\frac{1}{4} \pi 10^2 = 25\pi$. If he turns corner B, he can run around in an area that is one eighth of a circle of radius 10 which is $\frac{1}{8} \pi 10^2 = 12.5\pi$. Therefore, his total area is $128\pi + 25\pi + 12.5\pi = 165.5\pi$.

2. There is a square town of unknown dimensions. There is a gate in the middle of each side. Twenty paces outside the North Gate is a tree. If one leaves the town by the South Gate, walks 14 paces due south, then walks due west for 1775 paces, the tree will just come into view. What are the dimensions of the town? Show your steps. (Problem taken from the Chinese text *Seven Chapters on the Mathematical Art* ~200B.C.)

**Solution:**

From the picture above, we see by similar triangles, $\frac{20}{x/2} = \frac{x + 34}{1775}$. Cross-multiplying, we get $35500 = \frac{x^2}{2} + 17x$ or $x^2 + 34x - 71000 = 0$ which can be factored as $(x - 250)(x + 284) = 0$. Therefore, $x = 250$ and the dimensions of the town are 250 paces by 250 paces.

3. North Carolina wants to know how many 5-digit zip codes that can make under the following conditions

- a zip code can only use the digits 0, 1, 3, 4, 7, and 9
- no digit can be repeated
- zip codes must be between 35000 and 80000
- all zip codes must be even

How many different zip codes can North Carolina have? Show your steps.

**Solution:** If the first digit is a 3, the next digit must be either 7 or 9 and the last digit must be either 4 or 0. With each of those 4 possibilities, there are 3 more digits to choose the last 2 slots, which can be done 6 ways. Therefore, there are 24 zip codes if the first digit is 3.

If the first digit is 4, then the last digit must be 0 and the other 3 places can be any of the 4 remaining digits so there are 24 zip codes.

If the first digit is 7, the last digit is either 0 or 4 and the other 3 places can be any of the 4 remaining digits so there are 24 possibilities in each of the 2 cases for a total of 48.

So the number of zip codes is $24 + 24 + 48 = 96$. 