AN INTUITIVE ANALYSIS OF MAXIMAL SQUASHED FLAT ANTICHAINS

by

James Kolesar

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Approved by:

Katherine Mawhinney, Ph.D., Thesis Director

William J. Cook, Ph.D., Honors Director, Department of Mathematical Sciences

Eric S. Marland, Ph.D., Chair, Department of Mathematical Sciences

Abstract

Maximal squashed flat antichains are objects within algebraic mathematics whose concepts are deeply rooted in higher level algebra and combinatorics. This paper seeks to find an intuitive map behind the reasoning of these analyses, as well as provide a text that is an accessible introduction to the subject for undergraduate mathematics students, rather than more rigorous existing resources. The methodology behind this consists mainly of detailing my personal discovery of these proofs and their reasoning after learning about the basic characteristics of maximal squashed flat antichains, which will hopefully aid in highlighting the genuine intuition that powers this paper. The analysis itself focuses primarily on the marginal values of MSFA series, as well as the fractal-like nature of their structure.

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1 An Introduction to Maximal Squashed Flat Antichains

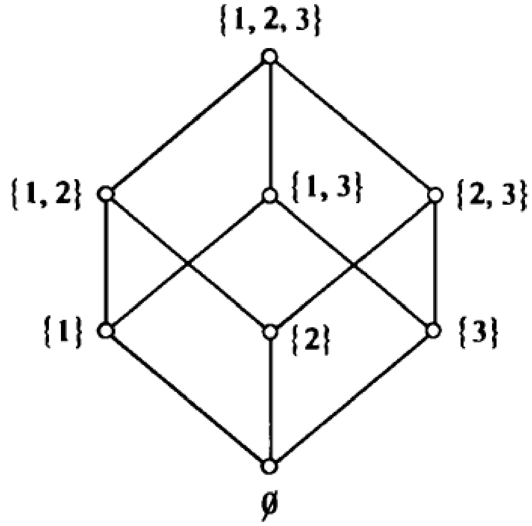
1.1 Introduction

This paper consists of a deep dive into maximal squashed flat antichains: their different qualities and characteristics from primarily a set theoretic perspective. In order to dive into these topics, however, there are many questions that need to be answered concerning MSFA's, first and foremost being what they are exactly. Let's get started by explaining each of the words making up the acronym, beginning with what exactly an antichain is.

1.2 Antichains

Within the world of set theoretics it is possible to have a collection of sets. Let's take sets containing integers, for example. We can define the set $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{7, 8, 9\}$. We can group these sets together into the collection $\mathcal{F}$, such that $\mathcal{F} = \{A, B, C\} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. When examining collections of sets, we can look towards the ***comparability*** of the different sets within that collection. With real numbers or integers we typically use $\leq$ and $\geq$ to compare values $(-1 \leq 2)$, but with sets we compare them using $\subseteq$. For example, $\{1, 2\} \subseteq \{1, 2, 3\}$. While it's not an issue we usually run into when comparing two real numbers, often times we can run into two sets which are incomparable. For example, $\{1,\ 2\} \nsubseteq \{3,\ 4\}$ and $\{3,\ 4\} \nsubseteq \{1,\ 2\}$. Tying this idea of compatibility back into collections of sets, if we wind up having a collection of sets such that all of the sets can be compared with each other it's called a ***chain***. For example, $\mathcal{H}$ = $\{\{2\},\ \{1,\ 2\},\ \{1,\ 2,\ 3\},\ \{1,\ 2,\ 3,\ 4\}\}$ or $\mathcal{I} = \{I_1, \ldots, I_n\}$ where $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$. Conversely, if none of the sets within our collection are comparable to each other, we call that collection an ***antichain***. The first collection of sets we looked at, $\mathcal{F}$, is an example of an antichain. Even though the number values increase between our sets, none of the sets are subsets of each other, making them all incomparable.

A common example of these set collections in mathematics are collections of subsets from a certain ***n-set***. An n-set is simply the set of positive integers from 1 to n, so a 3-set would be $\{1, 2, 3\}$, and its possible subsets would be $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. These subsets can be organized into a subset lattice which can help us visualize their comparability with each other:



Any two sets that wind up being comparable with each other have a direct, one-way path between the two of them. For example, $\{1\}$ can be connected to $\{1,2,3\}$ which shows that it's a subset of $\{1,2,3\}$.

This lattice makes up a good visualization of the subsets of the 3-set, however as n gets larger lattices become more and more complex (and less and less helpful) when they're illustrated. Understanding the idea behind subset lattices, however, will be a great asset for wrapping your head around MSFA's from a set theoretical perspective moving forward.

While the 3-set subset lattice is limited in its comparability, there are a few key ideas to be gleaned related to antichains. For one, simply taking the collection of every subset of a certain cardinality will result in an antichain: $\{\{1,2\},\{1,3\},\{2,3\}\}$ for example. This makes intuitive sense, as any two subsets of the same cardinality that don't have the exact same elements can't be subsets of each other, and this idea holds for any subset of size k, or a ***k-subsets***, of any n-set. Let's take the 3-subsets of the 5-set, for example. $\{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,5\}, \{1,3,5\}, \{2,3,5\}, \{1,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}\}$ is an antichain! Another takeaway from our 3-set lattice is the idea of introducing subsets of different cardinalities into our single level antichain collections. Let's say we wanted to cram $\{1\}$ into our 2-subset collection. We would in turn have to remove $\{1,2\}$ and $\{1,3\}$ from our 2-subset collection if we want to maintain its antichain status, leaving us with $\{\{1\},\{2,3\}\}$, since $\{1\}$ and $\{2,3\}$ aren't comparable.

1.3 Flat Antichains

This segues into another definition from our acronym. ***Flat*** antichains are collections of sets that are all incomparable with each other, and also consist of exclusively sets of cardinality k or (k$-$1). In other words, the cardinalities of all our sets need to be within 1 of each other. Looking at the 5-set for example, if our collection had only 2-subsets and 3-subsets in it, it'd be flat. Same for if it had only 3-subsets and 4-subsets, but collections with 2-subsets and 4-subsets would not be flat. The concept covered in the previous paragraph of starting with a collection of all the (k$-$1)-subsets and trying to force in k-subsets winds up being a crucial process when examining MSFAs, and in order to run through this process we need to talk about ***squashed ordering***.

1.4 Squashed Ordering and Squashed Flat Antichains

Typically, when looking at ordering a collection of k-subsets we think of lexicographic, or alphabetic, ordering. Similar to what we see in a dictionary, comparing two sets using lexicographic ordering will check the smallest value of both sets first to see which is lower, then the second smallest value if they're tied, then the third, and so on. The 3-subsets of the 5-set in lexicographical order, for example, looks like $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,3,5\}$, $\{2,4,5\}$, $\{3,4,5\}$. Squashed ordering, on the other hand, checks whatever value is largest amongst the values in our sets, then the next largest if they're tied, and so on. Looking at our previous example, $\{1,3,4\}$ would fall before $\{1,2,5\}$ in our squashed order, since the largest value in $\{1,3,4\}$ (i.e., 4) is lower than the largest value in $\{1,2,5\}$ (i.e., 5). The 3-subsets of the 5-set in squashed order are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,5\}$, $\{1,3,5\}$, $\{2,3,5\}$, $\{1,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$. A good go-to rule for listing subsets in squashed order is to check to see what the lowest value you can increase by one to give you a new set, then "reset" any elements smaller than the one you increased. Take $\{2,3,4\}$, for example. You can't increase 2 or 3 to get us a new set since elements must be unique in our 3-subsets, so you need to increase 4 to 5, which "resets" the first two elements to the smallest ones available: 1 and 2. Squashed ordering comes off as strange at first, and you might ask how this seemingly arbitrary ordering of sets plays a role in the construction of MSFAs. In order to illustrate the significance of squashed ordering, let's take a look at all of the potential (2, 3) flat antichains we can make in the 5-set (i.e., antichains containing elements exclusively of cardinality 2 or 3 drawn from the set $\{1,2,3,4,5\}$).

For (2, 3) flat antichains in the 5-set we can run through this process by hand by beginning with the complete collection of 2-subsets: $\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},$ $\{4,5\}\}$. Observant readers will note that this is a (2, 3) flat antichain: even though it doesn't include any 3-subsets, it still falls within the restrictions imposed by being a flat collection and an antichain. Even more observant readers will note that we can create many more flat antichains by simply removing any number of our 2-subsets, however this fact isn't particularly useful when it comes to the construction of MSFAs for reasons we'll get into shortly. Now we can force our first 3-subset (in squashed order: $\{1,2,3\}$) into our existing flat antichain, requiring the removal of necessary 2-subsets. This leaves us with $\{\{1,2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$. Repeating this process with the second 3-subset in squashed order gives us $\{\{1,2,3\},\{1,2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$. This process can be repeated for all ten of our 3-subsets, leaving us with this pattern, where the cardinality of each collection is noted to the right:

$\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \mid 10$

$\{\{1,2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \mid 8$

$\{\{1,2,3\},\{1,2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \mid 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \mid 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \mid 8$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{3,5\},\{4,5\}\} \mid 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{4,5\}\} \mid 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{4,5\}\} \mid 8$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\}\} \mid 8$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\}\} \mid 9$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\},\ \{3,4,5\}\} \mid 10$

1.5 Maximal Squashed Flat Antichains

With this pattern established there's one final stipulation to introduce for our MSFAs, that being the idea of ***maximal*** squashed flat antichains. Throughout the course of introducing 3-subsets into our collections, the removal of 2-subsets from our antichain was usually necessary, but we ran into a few occasions where we were able to add a 3-subset without needing to remove any of our 2-subsets (take going from $\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ to $\{\{1,2,3\},\{1,2,4\},\{1,3,4\},$ $\{2,3,4\},$ $\{1,5\},$ $\{2,5\},\{3,5\},\{4,5\}\}$: $\{2,3,4\}$ was added without removing any 2-subsets). A flat antichain in which no k-subsets can be added to our collection without abandoning the antichain property is called a maximal, while a flat antichain in which no (k-1)-subsets can be added into our collection while keeping it an antichain is a ***full*** antichain. In our squashed ordered series, all of our antichains will be full, while only some of them will be maximal. Due to this we can view full squashed flat antichains (FSFAs) as a more general version of MSFAs. By this definition we can determine which collections from our previous example were maximal and which were only full:

$\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 10 | Max.

$\{\{1,2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 8 | Max.

$\{\{1,2,3\},\{1,2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 7 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 7 | Full

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 8 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{3,5\},\{4,5\}\}$ | 7 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{4,5\}\}$ | 7 | Full

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{4,5\}\}$ | 8 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\}\}$ | 8 | Full

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\}\}$ | 9 | Full

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\},\{3,4,5\}\}$ | 10 | Max.

So by applying each restriction discussed so far we can come up with all of the possible (2, 3) MSFAs in the 5-set:

$\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 10 | Max.

$\{\{1,2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 8 | Max.

$\{\{1,2,3\},\{1,2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 7 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ | 8 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{3,5\},\{4,5\}\}$ | 7 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{4,5\}\}$ | 8 | Max.

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\},\{3,4,5\}\}$ | 10 | Max.

Thus concludes the introduction of MSFAs. There were a lot of concepts that were barely touched on throughout this section that will be explored in much more detail moving forward, and gaining a solid understanding of the basics will be very helpful in understanding this paper moving forward.

2 Marginal Value Analysis of MSFAs and FSFAs

2.1 Introduction

The majority of analysis regarding MSFA/FSFAs focuses on their quantifiable characteristics, primarily their various size functions, including weight, volume, and cardinality. The final example on that list, cardinality, is at the heart of marginal value analysis, and will be the primary focus of the remainder of this thesis, but other size functions will at least be touched on within the paper. With that being said, let's take a look at what exactly is meant by "marginal value" in relation to MSFAs.

2.2 Cardinality and Marginal Values of MSFAs/FSFAs

Thanks to the S in FSFA, the list of all possible FSFAs can be ordered, beginning with the set containing all (k−1)-subsets, ending with the set containing all k-subsets, and working forward by adding the next possible k-subset in squashed order. This opens up many possibilities for categorizing and generating all potential FSFAs for a given k and n in that it allows us to identify the patterns exhibited in general by FSFAs moving from our (k−1) collection to our k collection. At the crux of these patterns are the ***marginal values*** of our FSFAs: that being the change in size as we move from one FSFA to another in order, from start to finish. Let's take a look at the example we worked through earlier in part one, first examining the FSFAs:

$\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \ \mid\ 10$

$\{\{1,2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \ \mid\ 8$

$\{\{1,2,3\},\{1,2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \ \mid\ 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \ \mid\ 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\} \ \mid\ 8$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{3,5\},\{4,5\}\} \ \mid\ 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{4,5\}\} \ \mid\ 7$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{4,5\}\} \ \mid\ 8$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\}\} \ \mid\ 8$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\}\} \ \mid\ 9$

$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\},\{3,4,5\}\} \ \mid\ 10$

The initial size of our FSFA is pretty easy to determine: since we begin with all of the potential (k−1)-subsets from an n set, the size of our first FSFA is always $\binom{n}{k-1}$. In our case, $\binom{5}{2}$ = 10. We can find the next FSFA in our sequence by adding the first 3-subset in squashed order, $\{1, 2, 3\}$, which necessitates the removal of $\{1, 2\}, \{1, 3\}$, and $\{2, 3\}$ from our collection, meaning we've lost three 2-subsets but added one 3-subset, making our total loss 2. Throughout this paper, marginal values will be indicative of how many sets have been taken away from our collection, so losing two sets results in a marginal value of 2, not -2. Moving to the next FSFA, we see that adding $\{1, 2, 4\}$ removes $\{1, 4\}$ and $\{2, 4\}$ from the collection, giving us a marginal value of one and taking our total size 7. Continuing this process we notice the following series of marginal values from start to finish:

$$2, 1, 0, -1, 1, 0, -1, 0, -1, -1$$

At first glance we might notice a pattern emerge, one which might be even clearer if written as:

2, 1, 0, -1,

1, 0, -1,

0, -1,

-1

This is great news for us as it promises some sort of pattern for us to study. Let's take a look at another example and see if we can find some consistent similarities between the two, this time looking at the cardinalities of the (3, 4) flat under the 7 set:

$$3, 2, 1, 0, -1, 2, 1, 0, -1, 1, 0, -1, 0, -1, -1, 2, 1, 0, -1, 1, 0, -1, 0, -1, -1, 1, 0, -1, 0, -1, -1, 0, -1, -1, -1$$

Which we can rewrite as:

3, 2, 1, 0, -1,

2, 1, 0, -1,

1, 0, -1,

0, -1,

-1,

2, 1, 0, -1,

1, 0, -1,

0, -1,

-1,

1, 0, -1,

0, -1,

-1,

0, -1,

-1,

-1

As it turns out, we do wind up observing a similar pattern. The pattern continues for longer this time around, and it turns into something almost fractal-like as n and k get large enough for us to see a looping pattern emerge. What causes these marginal values to exhibit this pattern, and can we form some sort of general algorithm to determine what these patterns will be for any n and k?

2.3 Sets' Shadows and Marginal Values

Let's take a look at what exactly happens when we add one of our k-subsets to an FSFA full of (k-1)-subsets, specifically the first 3-subset in our (2, 3)-5 example. $\{1, 2, 3\}$ houses inside of it the three 2-subsets: $\{1, 2\}, \{1, 3\}, \{2, 3\}$. We notice the second 3-subset, $\{1, 2, 4\}$, also houses 3 subsets ($\{1, 2\}, \{1, 4\}, \{2, 4\}$). This makes intuitive sense, as any set should house $\binom{k}{k-1}$ subsets of size (k−1), which we know is just k (3, in our case.) The collection of (n-1) subsets that are contained in any set $\mathcal{A}$ of cardinality n is referred to the ***shadow*** of $\mathcal{A}$, denoted by $\nabla\mathcal{A} = \{\mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A} \text{ and } |\mathcal{B}| = |\mathcal{A}| - 1\}$. When looking at marginal values specifically we focus on the size of the ***new shadow*** of each k-subset we add to our collection: that being the (k-1)-subsets that are contained within our k-subset that aren't yet contained in any of the previous k-subsets in squashed order. In the previous examples in this paragraph, the new shadow of $\{1, 2, 3\}$ would be all three subsets contained in it, but $\{1, 2, 4\}$ would have a new shadow consisting of $\{1, 4\}$, and $\{2, 4\}$, since $\{1, 2\}$ is found in $\{1, 2, 3\}$ already. So what patterns do these new shadows follow? Let's look at our (3, 4)-7 example for guidance then extend a general proof of their pattern.

2.4 Discrete Attempts at a General Solution for Marginal Values

Let's ascribe each marginal value to the collection found before it in squashed order. This means our first marginal value of 3 is attached to the first collection in our example of all 3-subsets. Our first 4-subset $\{1, 2, 3, 4\}$ contains $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$, so it has a new shadow of size 4, which matches with our marginal value of 3 since we're removing 4 3-subsets and adding 1 4-subset. Moving down the line until the end of our first row we add $\{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}$, and $\{2, 3, 4, 5\}$, which have new shadows of size 3, 2, 1, and 0 respectively. The next row consists of adding $\{1, 2, 3, 6\}, \{1, 2, 4, 6\}, \{1, 3, 4, 6\}$ and $\{2, 3, 4, 6\}$, which have new shadows of size 3, 2, 1, and 0. The next row has us adding $\{1, 2, 5, 6\}, \{1, 3, 5, 6\}$, and $\{2, 3, 5, 6\}$, new shadows of size 2, 1, 0. Next we add $\{1, 4, 5, 6\}, \{2, 4, 5, 6\}$, giving us 1 and 0, and finally we add $\{3, 4, 5, 6\}$, which has a new shadow of size 0.

Using intuition we see that during our first row of marginal values, the cardinality of the new shadows moves from k to 0, meaning there are k+1 sets in total that are added within the first row. Thinking in squashed order, this will take us from the first k-subset $\{1, 2, 3, 4\}$ to the last k-subset that doesn't include k+2: $\{2, 3, 4, 5\}$. While this is clearly seen discretely, we can also find reasoning in combinatorics, since each k-subset including exclusively elements 1, 2, ..., (k+1) needs to exclude one value from the first k+1 elements of n, so there will always be exactly k+1 sets that only include values of 1 through (k+1). The next row of marginal values for our (3, 4)-7 FSFA series starts off with $\{1, 2, 3, 6\}$ and goes to $\{2, 3, 4, 6\}$, which makes sense when compared to our initial row. Every single 3-subset containing numbers exclusively from 1 through 5 has been covered by the first 5 4-subsets added, so any 3-subsets in our new shadows need to include 6. This essentially restricts the number of spaces we have available at our disposal, and now we're examining a reset version of the new shadow of $\{1, 2, 3\}$, appending 6 onto whatever values we come up with. This is why we find a new shadow of size 3. We do the same for our following 3-subset with 6 appended onto it, $\{1, 2, 4\}$, which has a new shadow of size 2 ($\{1, 2\}$ has already been expended by $\{1, 2, 3\}$), so the subsequent new shadow for $\{1, 2, 4, 6\}$ must also be 2, since we are appending 6 onto the each element of the new shadow of $\{1, 2, 4\}$. This pattern continues down the row, and suddenly the fractal-esque patterns we observed earlier are making complete sense. This row uses up all of the subsets involving 1, 2, 3, 4, and 6, meaning any more are going to need both 5 and 6 in order to be a part of the new shadow, which means we look towards

$\{1,2,5,6\}$, $\{1,3,5,6\}$, and $\{2,3,5,6\}$ to pull 1-subsets that we append both 5 and 6 to in order to find the new shadow of these sets, which we see through the first set having a new shadow of size 2 ($\{1,5,6\}$ and $\{2,5,6\}$, then $\{1,3,5,6\}$ has a new shadow of size 1, only having $\{3\}$ as a potential 1-subset that wasn't used in our last set. We see $\{2,3,5,6\}$ has an empty new shadow, since $\{2\}$ and $\{3\}$ have been used. The final 3-subset, $\{4,5,6\}$, is found in the 4-subset, $\{1,4,5,6\}$, then there are two more 4-subsets to finish out all of our potential 4-subsets that only have elements consisting of the numbers one through six.

This process is essentially repeated when we move into 4-subsets with seven as an element: we move through the 2-subsets involving elements one through four with seven appended onto them in the sixth line of our marginal values, then 1-subsets from the numbers of one through four appended onto $\{5,7\}$, and finally the 1-subsets from the numbers one through five appended onto $\{6,7\}$. That was a bit of a lengthy run through, but more than possible. Let's see if we can build off of this to start constructing a general pattern that works for FSFA's with any k and n.

The new shadow of the first k-subset we add to our (k−1) flat will always be of size k, and that k-subset will consist of the first k elements of n. The next (k−1) k-subsets we add will be running up the squashed order for our k-subsets, using up all the potential k-subsets containing the first (k+1) elements of n. We see this in our example, where the first 5 elements added to our FSFAs were the ones containing 1, 2, 3, 4, or 5. The size of the new shadows of these go from k to 0. The following (k−1) k-subsets we add will continue in squashed order, using up all of the possible (k−1)-subsets including the first k elements of n, appended to the (k+2)$^{\text{th}}$ element of n. In our example, the next 4 subsets were the potential 3-subsets from $\{1,2,3,4\}$ all with 6 appended onto them. Their new shadows go from k−1 to 0. The next (k−1) k-subsets will be comprised of the (k−2)-subsets from the first (k−1) elements of n appended to the (k+1) and (k+2)$^{\text{th}}$ elements of n. In our example, the 2-subsets of $\{1,2,3\}$ appended to $\{5,6\}$. They have new shadows of k−2 to 0. This pattern repeats itself down the line, where we take the (k−i)-subsets from the first (k−i+1) elements of n and append them to the set $\{(k+2),(k+1),\ldots,(k-i+3)\}$, for i from 0 to (k−1). This allows us to form a shape of our new shadows in a general sense, similar to how we did in our examples:

k, (k-1), (k-2), ... , 1, 0

(k-1), (k-2), ... , 1, 0

(k-2), . . . , 1, 0

. . .

0

Or, using marginal values rather than new shadows:

(k-1), (k-2), (k-3) ... , 0, -1

(k-2), (k-3), ... , 0, -1

(k-3), . . . , 0, -1

. . .

-1

This visualization makes seeing our patterns a bit easier. The very first k-subset we add to our FSFA covers all of the first k elements of n. The first row of marginal values represents all of the k-subsets it takes to add the first (k+1) elements of n into our FSFA. The entirety of the triangle is required to add all possible k-subsets containing exclusively the first (k+2) elements of n. So what happens when we want to add the $(k+3)^{th}$ element of n into our k-subsets? Well, we know that our new shadow will be (k-1), and working up the k-subsets in squashed order we essentially repeat our previous process but with one fewer space where elements can go, since (k+3) is occupying one of our spots. This means we start with:

k, (k-1), (k-2), ... , 1, 0

(k-1), (k-2), ... , 1, 0

(k-2), ... , 1, 0

...

0

(k-1), (k-2), ... , 1, 0

(k-2), ... , 1, 0

...

0

(k-2), ... , 1, 0

...

0

...

...

0

Where the second triangle represents running through the (k-1)-subsets containing only the first (k+1) elements of n appended with (k+3), the third triangle represents running through the (k-2)-subsets containing only the first k elements of n appended with (k+3) and (k+2), and so on until we reach the subset consisting of (k+3), (k+2), ..., 5, 4 appended onto the 0-subset of the first 2 elements of n.

2.5 A Combinatorial Outlook on our General Solution

This process of intuitive proof begins to get very tedious as we move past this point (as if it wasn't already) so let's use what we've gleaned so far to produce some sort of less painful combinatoric proof. What do we have so far: the width and height of our triangles are based on how large k is, as well as the number of triangles necessary to complete each additional element of n. The number of times that we iterate on n, however, is dependent on (n-k), as (n-k) represents the number of elements we'll need to introduce past our initial k-subset (the first k-subset in squashed order.) We also have a general solution for the first 3 iterations of any FSFA marginal value sequence, we just need a general solution to what future iterations will look like and we'll have our proof.

Purely going off of pattern recognition, it seems like on each iteration, we repeat the previous iteration starting at (k-1), then repeat it again starting with (k-2), (k-3), all the way down to starting with 0. This means that to add all subsets containing the $(k+4)^{\text{th}}$ element of n we would need to run through our sequence of triangles again, beginning with (k-1), then the full sequence a second time beginning with (k-2), and so on, until we get a full sequence of triangles that would begin with 0, which would of course just be 0.

With this pattern as an initial "end goal", we took the liberty to develop a general algorithm for mapping out these marginal value and size sequences in Python to visualize a few of the concepts surrounding marginal value analysis on a larger scale. For starters, we need to figure out exactly what we want Python to do in order to calculate a pattern of marginal values that matches our hypothesis. What we're looking for is a series of for loops that causes us to run through the values of k to -1, then repeat that run through from (k−1) to -1, finish that repetition until we begin with -1, then repeat everything we've done so far multiple times, with starting values ranging from (k−1) to -1, and keep going for (n−k) iterations. That's exactly what a nested for loop does, so in terms of rudimentary code we're looking for something that does this:

```python
for j in range(4, -1, -1):
  for i in range(j, -1, -1):
    for g in range(i, -1, -1):
      for l in range(g, -1, -1):
        for h in range(l, -1, -1):
          print(h)
```

For a general code chunk, we're looking for one that will begin at k and repeat (n−k) times. This will require some recursive code that essentially creates the nested for loops from before, except it prints the value of whatever variable is being iterated through on the loop:

```python
def FSFAMV(k, t):
  for j in range(k, -1, -1):
    if(t > 1):
      FSFAMV(j, t-1)
      if(j != 0):
        print("")
    if(t == 1):
      print("%d, " %(j-1), end="")

def printFSFAMV(n, k):
  print("")
  print("Marginal values for the", k, "|", k-1, "FSFAs under", n)
  t = n-k
  FSFAMV(k, t)
```

This code also takes the liberty of printing the values in the triangular format that's been shown throughout the previous examples in the paper. Below is the console that results from calling the function with $n = 7$ and $k = 4$:

```
Marginal values for the 4 | 3 FSFAs under 7
3, 2, 1, 0, -1,
2, 1, 0, -1,
1, 0, -1,
0, -1,
-1,
2, 1, 0, -1,
1, 0, -1,
0, -1,
-1,
1, 0, -1,
0, -1,
-1,
0, -1,
-1,
-1,
```

Our code looks promising so far, as it follows exactly what we've seen in our discrete examples, which we know to be true just based off of running through the process by hand. Let's look at what happens when we plug $n = 8$ and $k = 4$ into our algorithm:

```
Marginal values
3, 2, 1, 0, -1,
2, 1, 0, -1,
1, 0, -1,
0, -1,
-1,
2, 1, 0, -1,
1, 0, -1,
0, -1,
-1,
1, 0, -1,
0, -1,
-1,
0, -1,
-1,
-1,
```

```
2, 1, 0, -1,
1, 0, -1,
0, -1,
-1,
1, 0, -1,
0, -1,
-1,
0, -1,
-1,
-1,
1, 0, -1,
0, -1,
-1,
0, -1,
-1,
-1,
0, -1,
-1,
-1,
-1,
```

The good news is our code is doing exactly what we wanted it to be doing. With (n−k) increasing in value by one, we added one more iteration onto our repetitive process, taking the marginal values that we displayed throughout our first k+3 elements and running through them once with a new shadow starting at (k−1) (meaning our marginal values begin at (k−2), which in this case is 2,) again starting with a marginal value of (k−3), all the way until we began the entire process with −1. We still don't know why this is the case, however, and we aren't necessarily sure that it's even correct yet. Once thing we can do to help verify that this pattern is a proper end goal before moving on is run code that displays the cardinalities of our FSFAs from start to finish based on the marginal values our previous code is generating. If the marginal value pattern is correct, the sizes should run from $\binom{n}{k-1}$ to $\binom{n}{k}$, since we know the first FSFA in our series is every single (k−1)-subset in the n-set, and the final FSFA in our series is every single k-subset in the n-set. Here's the code that uses the same marginal value function to generate cardinalities:

```
def FSFAsizes(k, t):
  global IV
  for j in range(k, -1, -1):
    if(t > 1):
      FSFAsizes(j, t-1)
      if(j != 0):
        print("")
    if(t == 1):
      IV = (IV-j+1)
      print("%d, " %IV, end="")

def printFSFAsizes(n, k):
  print("")
  print("Sizes for the", k, "|", k-1, "FSFAs under", n)
  t = n-k
  print("%d," %(comb(n,k-1)))
  FSFAsizes(k, t)
```

It takes an initial value of $\binom{n}{k-1}$ and adds to that the marginal value generated by out nested for loop. Here is what prints to the console for $n = 7$ and $k = 4$:

```
Sizes for the 4 | 3 FSFAs under 7
35,
32, 30, 29, 29, 30,
28, 27, 27, 28,
27, 27, 28,
28, 29,
30,
28, 27, 27, 28,
27, 27, 28,
28, 29,
30,
29, 29, 30,
30, 31,
32,
32, 33,
34,
35,
```

Here we see the cardinalities working through the (3, 4) FSFAs under the 7-set, starting with $\binom{7}{3} = 35$ and ending with $\binom{7}{4} = 35$. We see a similar pattern for the (3, 4) FSFAs under the 8-set:

```
Sizes for the 4 | 3 FSFAs under 8
56,
53, 51, 50, 50, 51,
49, 48, 48, 49,
48, 48, 49,
49, 50,
51,
49, 48, 48, 49,
48, 48, 49,
49, 50,
51,
50, 50, 51,
51, 52,
53,
53, 54,
55,
56,
```

```
54, 53, 53, 54,
53, 53, 54,
54, 55,
56,
55, 55, 56,
56, 57,
58,
58, 59,
60,
61,
60, 60, 61,
61, 62,
63,
63, 64,
65,
66,
66, 67,
68,
69,
70,
```

For these cardinalities we begin with $\binom{8}{3} = 56$ and we end with $\binom{8}{4} = 70$, which tells us that we're at least most likely to be on the right track in terms of this intuitive pattern we decided on earlier in the paper.

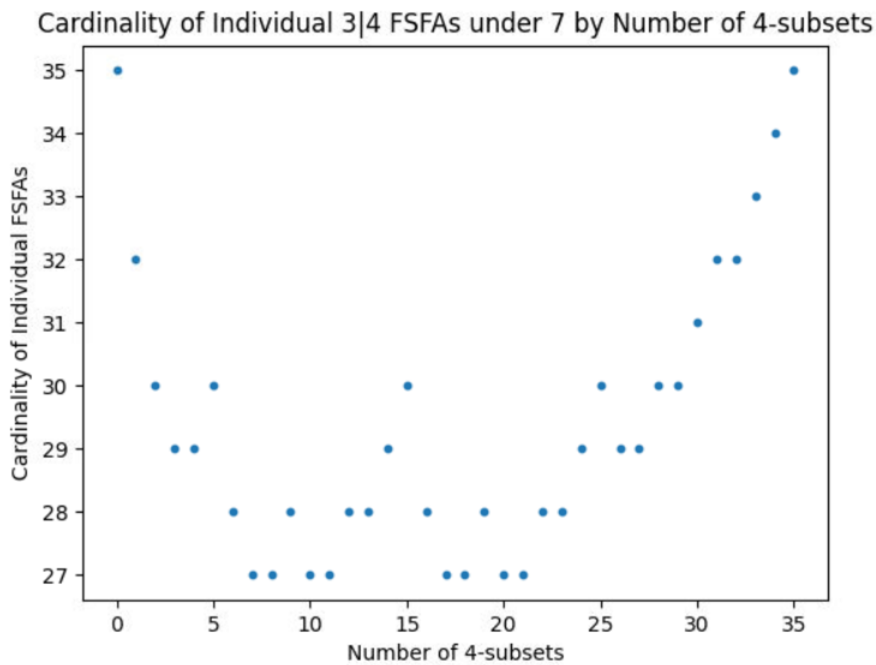
The results from our code were promising so far, but in order to aid in future exploration that would prove necessary in discovering a more solid reasoning for our pattern we need to plot the FSFA cardinalities. The original motivation to code the plotting capabilities came simply from a desire to better display the cardinalities for our FSFA series in the console, but that decision would up being crucial to find the missing piece in our marginal value analysis. In order to get there, however, we needed to write code to compile the cardinalities of our FSFAs into a vector:

```
def vectorFSFAsizes(k, t, a):
  global f
  global IV
  for j in range(k, -1, -1):
    if(t > 1):
      vectorFSFAsizes(j, t-1,a)
    if(t == 1):
      IV = (IV-j+1)
      a[f + 1] = IV
      f = f+1
  if(a[comb((k+t),k)] == comb((k+t),k)):
    return(a)
```

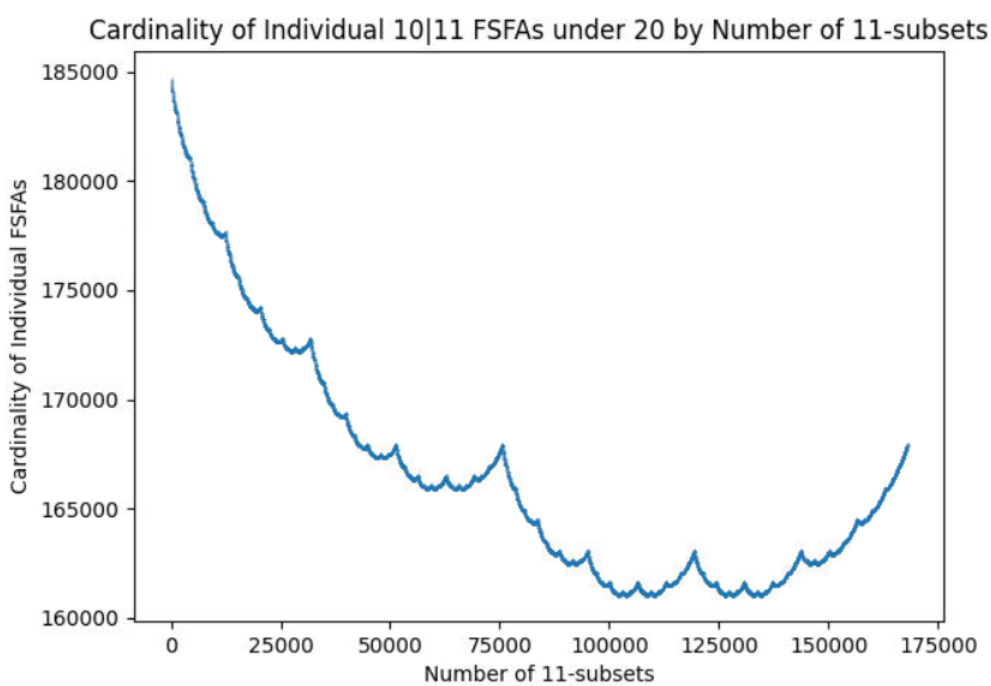
Then we need to plot that vector against a simple vector counting up from 0 to represent the number of k-subsets we've added to our FSFA at that point in the series:

```
def plotFSFAsizes(n, k):
  global dotSize
  a = returnVectorFSFAsizes(n, k)
  b = [0]*(comb(n,k)+1)
  for i in range(comb(n,k)+1):
    b[i] = i
  plt.scatter(b, a, s = dotSize)
  plt.title("Cardinality of Individual "+str(k-1)+"|
  "+str(k)+" FSFAs under "+str(n)+" by Number of "+k
  +"-subsets")
  plt.xlabel("Number of "+k+"-subsets")
  plt.ylabel("Cardinality of Individual FSFAs")
  plt.show()
```

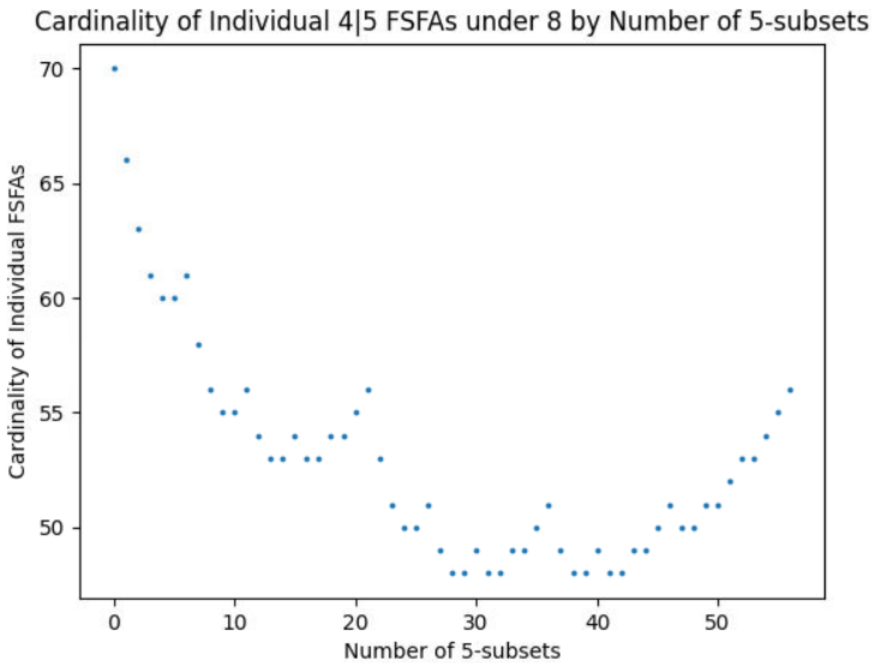
Plugging in $n = 7$ and $k = 4$ results in this plot:



This plotting method also gets rid of any potential limitations on the sizes of n and k, as it allows us to print every single cardinality on a set canvas, even if the FSFA series has over 100,000 FSFAs in it. Here's what we see when we set $n = 20$ and $k = 11$:



Looking at the patterns exhibited by these cardinalities at a large scale makes it ever so clear that there's some sort of fractal-like behavior existing within these series. This is still observed even when looking at smaller, less looming n's and k's. Take the (4, 5) FSFAs under 8:



We can clearly see the (3, 4) FSFA series of marginal values under 7 at the back half of this plot, and as it turns out the first half of the plot follows the pattern of the (4, 5) FSFA series of marginal values under 7. This observation is key, because it narrows down exactly what is happening to cause these repetitive patterns in our FSFA marginal values. Using the information gleamed from this we can construct a combinatorial proof as to how our FSFA marginal value series can be split into those of different k and n.

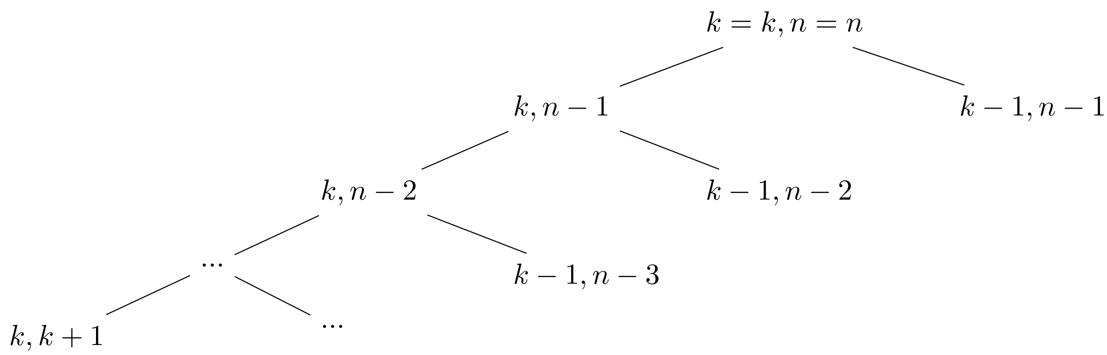
2.6 Marginal Values Series within Marginal Value Series

Using patterns we picked up from the plotting, we can claim that for any ((k−1), k) FSFA series under n, its marginal values can be represented by the marginal values of the ((k−1), k) FSFAs under (n−1) concatenated with the marginal values of the ((k−2), (k−1)) FSFAs under (n−1).

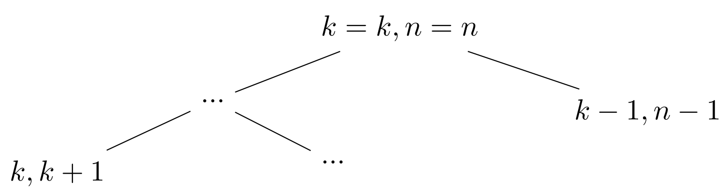
The proof of this is relatively simple and is rooted in binomial algebra: the marginal values of the ((k−1), k) FSFAs under (n−1) are equal to the marginal values displayed by the ((k−1), k) FSFA series under n for all collections before our nth element is introduced in squashed order, while the marginal values of the ((k−2), (k−1)) FSFAs under (n−1) mimic the marginal values for all future collections in our series (ones including the nth element). The reasoning behind this is found in the new shadows of these FSFA series. The new shadows for our k-subsets containing the elements of 1 through (n−1) will be the same regardless of n, as sets squashed order will not change based on the number of elements that are selected from to make up your sets. Once we add n into our pool of potential elements, every (k−1)-subset in the new shadow without n in it has already been accounted for. This means that all future subsets in our new shadow need to include n. We also know that no (k−1)-subsets in our new shadow have n in them, so any (k−1)-subset containing n will be apart of our new shadow. This essentially resets our new shadow for the first (n−1) elements, but it now has one less spot than before, meaning our new shadow is pulling (k−2)-subsets (with n appended to them) from (k−1)-subsets (with n appended to them) that are under the (n−1)-set, ((k−2), (k−1)) FSFAs under (n−1).

This is a general proof, which means these FSFAs can be broken down multiple times. Take our (4, 5) FSFA series under 8 for example. We've already noted that it can be broken down into the (4, 5) FSFA series under 7 and the (3, 4) FSFA series under 7, but both of those can be broken down into the (4, 5) FSFA series under 6, the (3, 4) FSFA series under 6, the (3, 4) FSFA series under 6, and the (2, 3) FSFA series under 6. We can keep breaking these down as many times as we want, however if we reach the point where $k = 0$ or $n = k$ our FSFA series only have a single collection within them, which essentially sets us back to what we were doing originally working through discrete examples. Using this method, however, we can show that the initial pattern we postulated that our marginal values would exhibit regardless of k and n is exactly correct. The important "stopping point" values for n and k are when $k = (n-1)$ and $k = 1$. The reason for these values is that the marginal value pattern for any $((n-2), (n-1))$ FSFA series under n will be one row of our larger triangular conscription of marginal values, starting with $(n-2)$ and going to -1, which was shown to be true discretely in section 2.4, and any $(0, 1)$ FSFA series under n will result in a marginal value pattern of 0 followed by $(n-1)$ -1's. Keeping that in mind we can construct a general lattice that splits any ((k−1), k) FSFA series under n into these important FSFA series to provide a proof to the general pattern for its marginal values.

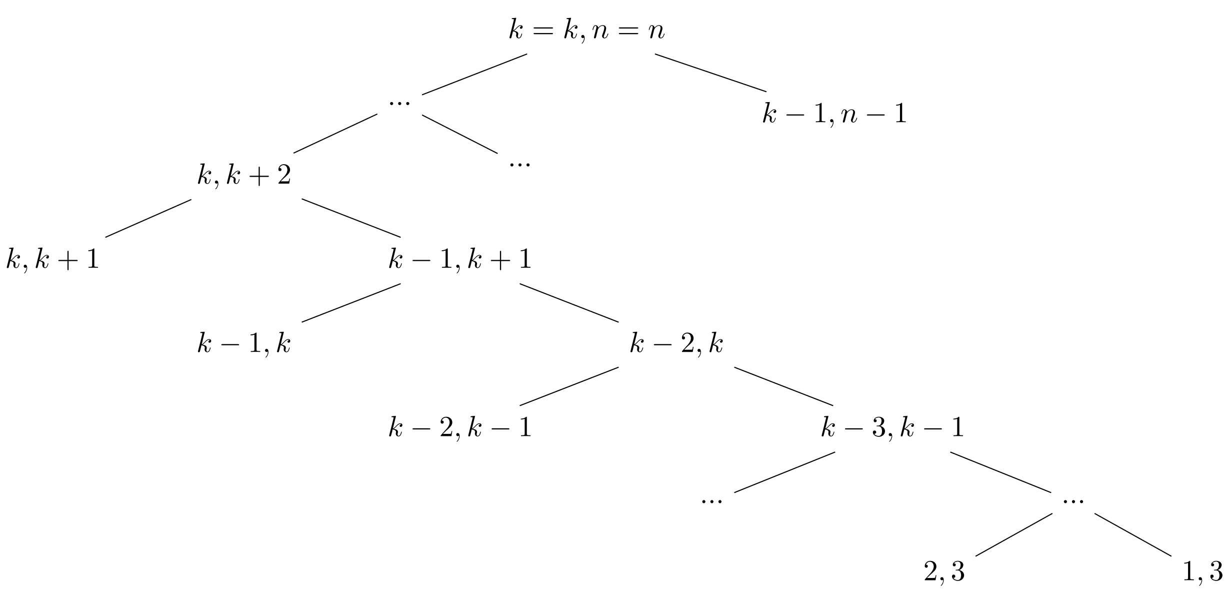
We can start with the ((k−1), k) FSFA series under n, which splits into the ((k−1), k) FSFAs under (n−1) concatenated with the marginal values of the ((k−2), (k−1)) FSFAs under (n−1). Next, the left node can be split into the ((k−1), k) FSFAs under (n−2) concatenated with the marginal values of the ((k−2), (k−1)) FSFAs under (n−2). The left node can continue to be split, until we wind up with the ((k−1), k) FSFAs under (k+1) on our left and ((k−2), (k−1)) FSFAs under (k+1) on the right:



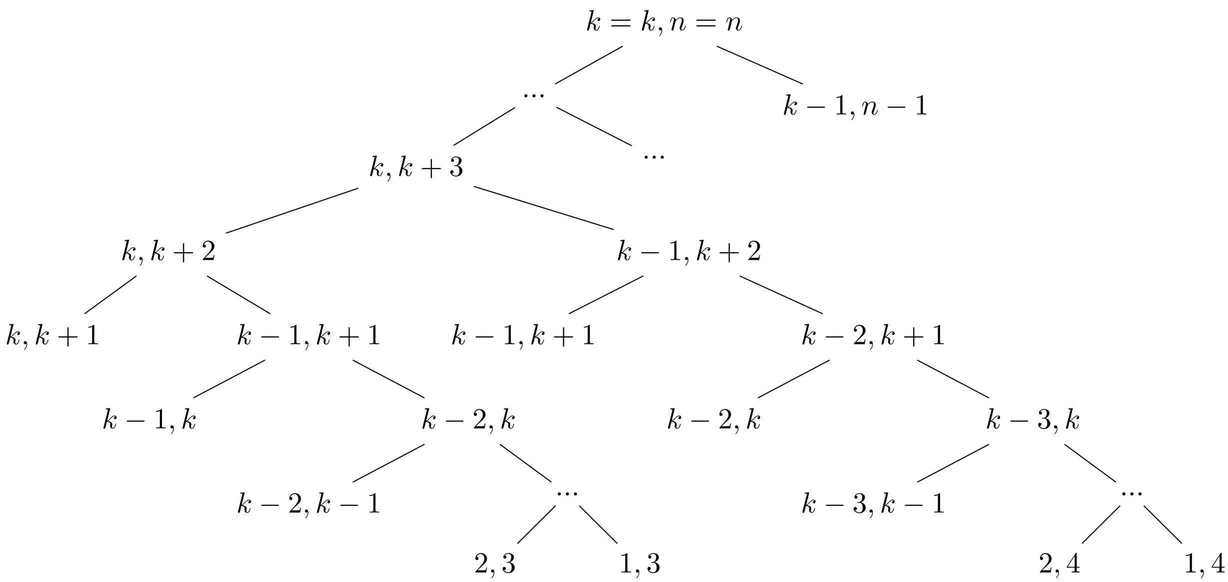
Which can be simplified for now as:



From here, instead of trying to work down from our original FSFA series, we can work up from our (k−1, k) under (k+1) FSFA series in a general sense. We know (k−1, k) under (k+1) must have come from (k−1, k) under (k+2), which also would split into (k−2, k−1) under (k+1). That FSFA series splits as well, and if we continue splitting down we wind up with:



The key things to notice about the separate FSFA series within our lattice is the distance between n and k, and the size of k itself. We already discussed what the marginal values of our series look like when $n = k + 1$, and now our lattice reveals what it looks like when $n = k + 2$. We wind up splitting down the line, forming the rows of our triangular representation of the marginal values starting with a new shadow of k and working down to beginning with a new shadow of zero. Extending this pattern up one more time gives us:



Because we've already worked out exactly what the $((k-2),(k-1))$ flat under $(k+1)$ in terms of our "simplest" FSFA marginal value series ($n = k+1$) we don't need to worry about expand that out. Same goes for the rest of our left nodes working down the $((k-2),(k-1))$ flat under $(k+2)$. Taking a look at what we find, we work through our initial triangle starting with a new shadow of cardinality $(k-1)$, then work through the initial triangle again starting with a new shadow of cardinality $(k-2)$, and continuing down until we recreate the initial triangle with a starting new shadow of cardinality 0. This mirrors the pattern we initially established perfectly, and looking up one more iteration we see the exact same pattern emerge, this time splitting into the $(k-2, k-1)$ flat under $(k+2)$ and working down to the (0,1) flat under 5. Having the distance between n and k being 3 in each of our iterations means each FSFA marginal value series indicates us working all the way down a repeating triangle structure, where each individual triangle represents a full FSFA marginal value series where $n - k = 2$. This pattern repeats up all the way until we get to any k and n we need to get to, thus proving the combinatorics our general pattern for the marginal values of any FSFA series.

2.7 Marginal Values with Exclusively MSFAs

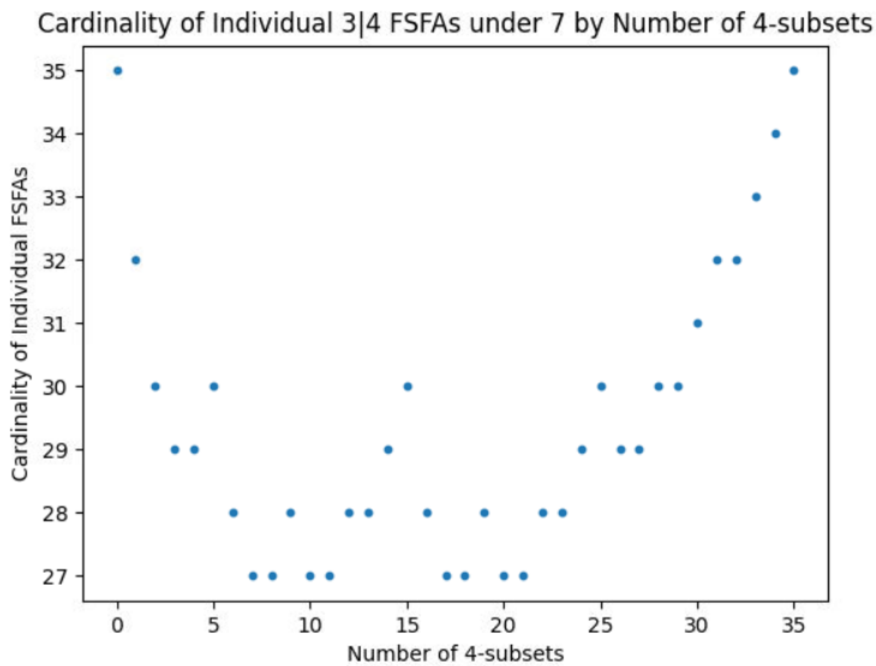
We've been spending a lot of time looking at the marginal values of FSFA series, which tend to be the more general version of our SFA series. Marginal values for maximal squashed flat antichains have special properties of their own that we can glean from the previous plots and proofs we've applied to FSFAs. First and foremost we can tweak our code a bit to write the cardinalities of only the MSFAs of our series into a vector, rather than all possible FSFAs:

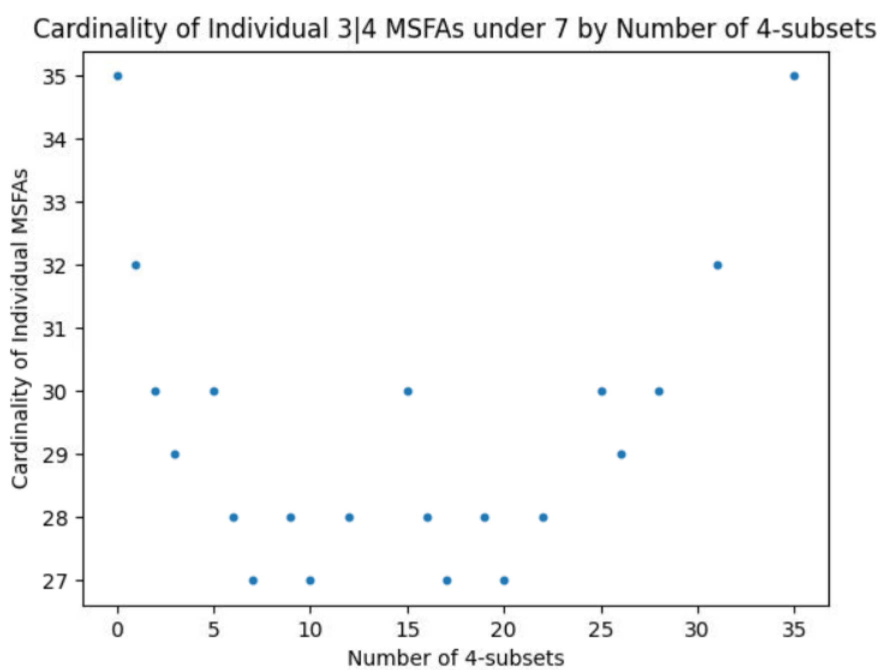
```
def returnVectorMSFAsizes(n,k):
    global vector
    global vector2
    unwanted = [None]*comb(n,k)
    for i in range(comb(n,k)):
        if(vector[i] == 0):
            unwanted[i] = i
    for i in range(comb(n,k)-1, -1, -1):
        if(unwanted[i] == None):
            del unwanted[i]
    for ele in sorted(unwanted, reverse = True):
        del vector2[ele]
    return(vector2)
```

Then we can take that vector and plot it in the exact same way we were plotting our FSFAs:

```
def plotMSFAsizes(n, k):
    global vector
    global dotSize
    a = returnVectorMSFAsizes(n, k)
    b = [0]*(comb(n,k)+1)
    for i in range(comb(n,k)+1):
        b[i] = i
    unwanted = [None]*comb(n,k)
    for i in range(comb(n,k)):
        if(vector[i] == 0):
            unwanted[i] = i
    for i in range(comb(n,k)-1, -1, -1):
        if(unwanted[i] == None):
            del unwanted[i]
    for ele in sorted(unwanted, reverse = True):
        del b[ele]
    plt.scatter(b, a, s = dotSize)
    plt.title("Cardinality of Individual "+str(k-1)+"|"+str(k)+" MSFAs under "+str(n)+" by Number of "+k+"-subsets")
    plt.xlabel("Number of "+k+"-subsets")
    plt.ylabel("Cardinality of Individual MSFAs")
    plt.show()
```

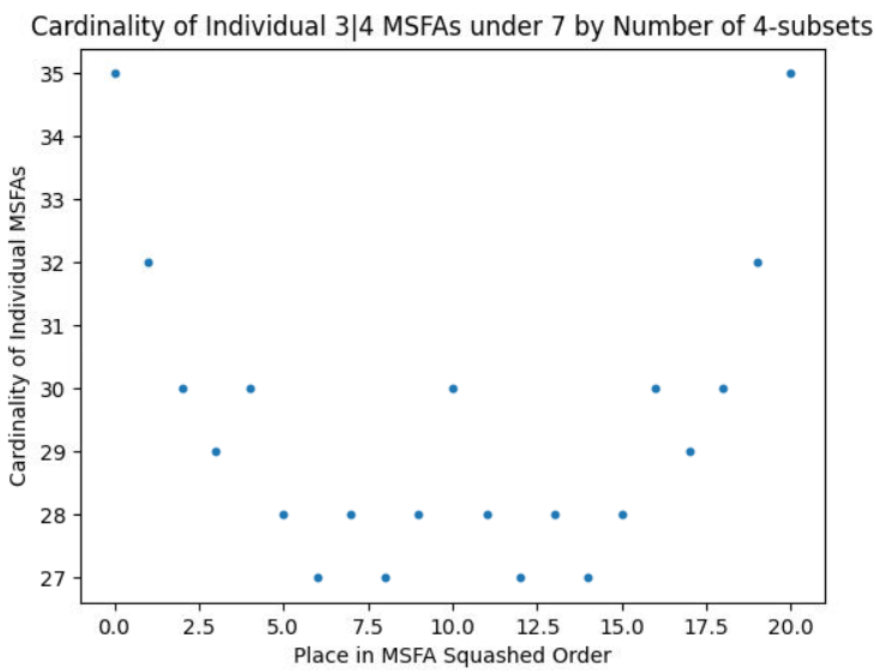
Let's take a look at the FSFA plot vs the MSFA plot when we plug 4 in for k and 7 in for n:



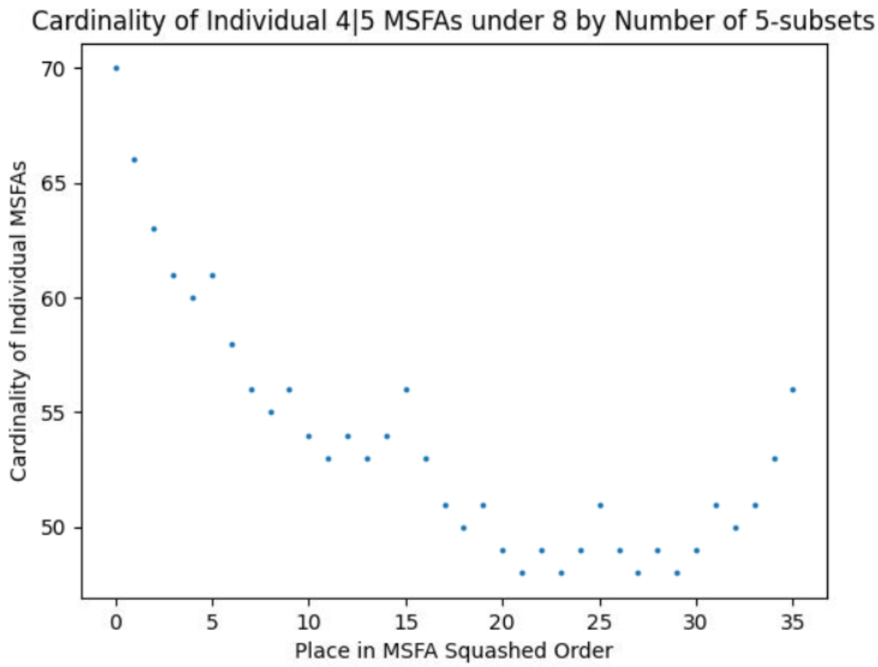


As we can see, the FSFA plot is close to a symmetric W shape, but it's a bit back-loaded in terms of its points. The MSFA plot, on the other hand, is fully symmetric in terms of both cardinalities and marginal values. The distance between the points aren't uniform, however, since our code is simply removing any non-maximal FSFAs. The symmetry is even clearer if we tweak our code to unify the distance between each of our points:

```
def plotMSFAsizesUnskewed(n, k):
  global vector
  global dotSize
  a = returnVectorMSFAsizes(n, k)
  b = [0]*(len(a))
  for i in range(len(a)):
    b[i] = i
  plt.scatter(b, a, s = dotSize)
  plt.title("Cardinality of Individual "+str(k-1)+"|"+str(k)+" MSFAs under "+str(n)+" by Number of k-subsets")
  plt.xlabel("Place in MSFA Squashed Order")
  plt.ylabel("Cardinality of Individual MSFAs")
  plt.show()
```



Here we can see the full symmetry of our (3, 4) MSFA series under 7 on display. We can also take a look at the (4, 5) MSFA series under 8 and compare it to the FSFA series we spent so much time analyzing earlier on in the paper:



Here we see that the fractal nature of the structure of our series remains consistent even when examining MSFAs rather than FSFAs, which makes sense considering the way new shadows work doesn't change when we look at exclusively maximal SFAs. That does raise an interesting question, however: if the FSFA series are asymmetrical, why is the MSFA series symmetrical? In addition to that, what MSFAs are symmetrical? As it turns out, the ((k−1), k) MSFAs under n plot will always be perfectly symmetrical to the ((n−k), (n−k+1)) MSFAs under n for any k and n, regardless of how different their FSFA plots are.

The proof behind this concept stems from the notion of a ***shade***, which is the reverse of a shadow, where the shade represents all sets $\mathcal{B}$ of size (k+1) that our initial set $\mathcal{A}$ is a subset of. It's denoted as: $\Delta\mathcal{A} = \{\mathcal{B} \mid \mathcal{A} \subseteq \mathcal{B}, |\mathcal{B}| = |\mathcal{A}| + 1\}$. The ***new shade*** functions very similarly to the new shadow, the two are almost complimentary in nature.

If we consider the first k-subset in squashed order and the last (n−k)-subset in squashed order, we notice that the two are compliments of each other. The same is true for the second k-subset in squashed order and the second to last (n−k)-subset in squashed order, and this continues all the way through our series. The new shadow of our k-subset and the new shade of its compliment will always be directly related, in that for all elements within our k-subset, there will be a corresponding element in its compliment such that the two elements are compliments themselves, and vice versa. The new shade of a k-subset must exclude all elements in its compliment, as well as exclude a single element from itself. The new shadow of an (n−k)-subset must include all of its own elements, as well as one from its compliment.

From this relationship we know that the cardinality of both the new shadow and new shade of its compliment must be equal. The cardinality of a new shadow is equivalent to the marginal value of the corresponding FSFA plus one, and the cardinality of a new shade when working in reverse squashed order indicates the amount of (n−k+1)-subsets that can be added without needing to remove any (n−k)-subsets minus one, which directly correlates with the amount of NMFSFAs found between two MSFAs within our series. Since these two cardinalities are equal and we know moving forwards in squashed order will result in the compliment moving backwards in squashed order, the size of the marginal value for our ((k−1), k) FSFA series will correspond directly with the number of NMFSFAs between the analogous compliments. This means that the marginal values will be the same moving forwards through our ((k−1), k) MSFAs as moving backwards through our ((n−k), (n−k+1)) MSFAs, and since we can substitute k for (n−k+1) to get the same compliments but with reversed roles, we know this idea holds both ways, proving our symmetry.

With this concept proven, we can posit the following lemma: the MSFA series for any n and k such that $n = 2*k+1$ will be symmetric. We know that the marginal values of the ((k−1), k) MSFA series under $n{=}2*k-1$ can be broken up into the marginal values of the ((k−1), k) MSFA series under $n{=}2*k-2$ and the marginal values of the ((k−2), (k−1)) MSFA series under $n{=}2*k-2$. These two are symmetric in nature, since since substituting $2*k-2$ for n in our ((n−k), (n−k+1)) gives us ((k−2), (k−1)) as the MSFA series that is symmetric to ((k−1), k). Since the ((k−1), k) MSFA series under $n{=}2*k-1$ can be broken into two symmetric parts, it itself is symmetric.

3 Other Weight Functions in Relation to MSFAs and FSFAs

3.1 Introduction

As mentioned in the previous section, while the cardinality of an MSFA remains the primary focus of this paper, there are a few other weight functions that can be applied to MSFAs or FSFAs that have been been studied by others in the field. The use and citation of outside sources has been purposefully avoided throughout the course of this paper, as the primary goal was to attempt to solve certain problems related to marginal value analysis essentially from scratch. The remaining pages, however, will be relying on citing other papers in order to provide context for some of the other ideas looked at when examining MSFAs, as well as create a road map to other research if any readers are interested in further reading on the subject after this (somewhat) introductory work.

3.2 General Definitions and Theorems Related to Size Functions of FSFAs

Before beginning there are a few definitions for weight functions that would be beneficial to go over. The ***volume*** of an antichain is the total number of elements found throughout every set in the collection. It can be denoted as $V(\mathcal{F}) = \sum_{F \in \mathcal{F}} |F|$ [2]. The ***BLYM*** (Bollobas-Lubell-Yamamoto-Meshalkin) ***value*** of an antichain is defined as $BLYM(\mathcal{F}) = \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|}$ [3]. The BLYM value of an antichain is actually integral in solving the theorem at the center of marginal value analysis; Sperner's Theorem.

Sperner's Theorem states that the largest possible cardinality of any antichain is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, and the maximum cardinality only occurs in collections consisting of every k-subset for $k = \left\lfloor \frac{n}{2} \right\rfloor$ or $k = \left\lceil \frac{n}{2} \right\rceil$ [3]. For example, largest possible cardinality of an MSFA under 7-set is 35, and the only MSFAs with that cardinality are the sets containing all of the 3-subset or all of the 4-subsets.

From our BLYM values we find the ***BLYM inequality***, which states that for any antichain has a BLYM value less than one. In other words, for any antichain $\mathcal{F}$, $BLYM(\mathcal{F}) = \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|} \leq 1$ [3].

3.3 A Brief Proof of Sperner's Theorem

We know that for any n and k such that $0 \le k \le n$, $\binom{n}{\lfloor \frac{n}{2} \rfloor} \ge \binom{n}{k}$. If we consider an antichain pulling from the n-set, defined as $\mathcal{S}$ we can say that s_k represents the total number of k-subsets within that antichain. Using the previous inequality we have $s_k / \binom{n}{\lfloor \frac{n}{2} \rfloor} \le s_k / \binom{n}{k}$. Next we can use the BLYM inequality by summing both sides of the inequality we established already in our proof to give us $\sum_{k=0}^{n} s_k / \binom{n}{\lfloor \frac{n}{2} \rfloor} \le \sum_{k=0}^{n} s_k / \binom{n}{k} \le 1$. This gives us $|\mathcal{S}| = \sum_{k=0}^{n} s_k \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$. We also know in order for the equality to hold, $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{k}$, which only happens when $k = \left\lfloor \frac{n}{2} \right\rfloor$ or $k = \left\lceil \frac{n}{2} \right\rceil$, thus proving Sperner's Theorem [3].

Sperner's Theorem is definitely something we'd like to put some more time into relating to marginal value analysis. The idea of analyzing the patterns of carinalities seems like it could be a very viable approach if attempting to find a combinatorial based proof of Sperner's Theorem and will certainly be the next thing looked into if any more time gets devoted to this personal research in the future. For a much more detailed look at Sperner's Theorem, K. Engel's Sperner Theory is an entire book dedicated to the analysis of sizes and cardinalities of finite sets, with antichains (also known as Sperner Families) being a primary focus [1].

3.4 Other Size Functions, Theorems, and Related Resource Resources

Relating to volume of our antichains, the ***Flat Antichain Theorem*** (FLAT) states that for any antichain $\mathcal{B}$ under the n-set, there is a flat antichain $\mathcal{F}$ that has the same volume and cardinality, so $|\mathcal{B}| = |\mathcal{F}|$ and $V(\mathcal{B}) = V(\mathcal{F})$. As an example, under the 6-set we can look at the antichain $\{\{1,2\},\{2,3,4,5\},$ $\{1,3,5,6\}\}$, which has a volume of 10 and cardinality of 3. A corresponding flat antichain would be $\{\{1,2,3\},\{2,3,4\},\{3,4,5,6\}\}$. FLAT is used in minimal weight analysis, examining general weight functions applied to MSFAs, and can be read more about in Full and Maximal Squashed Flat Antichains of Minimum Weight, by Griggs et al. [2].

A deeper analysis of new shades and new shadows can be found in Maximal Antichains of Subsets I: The Shadow Spectrum by Griggs et al. [4]. The binomial aspect of MSFAs is explored much more in Maximal Antichains of Subsets II: Constructions by Griggs et al. [5].

References

[1] K. Engel, *Sperner Theory*, Cambridge University Press, January 1997.

[2] J. Griggs, S. Hartmann, T. Kalinowski, U. Leck, and I. Roberts, *Full and Maximal Squashed Fat Antichains of Minimum Weight* (September 12, 2018).

[3] L. Babai and N. Raghunathan, *Sperner's Theorem and Bollobas Inequality*, `http://people.cs.uchicago.edu/~laci/REU05/potp/Jul20/notes-0720.pdf`. Accessed December 3, 2021.

[4] J. Griggs, S. Hartmann, T. Kalinowski, U. Leck, and I. Roberts, *Maximal Antichains of Subsets I: The Shadow Spectrum* (June 2021).

[5] J. Griggs, S. Hartmann, T. Kalinowski, and U. Leck, *Maximal Antichains of Subsets I: Constructions* (June 2021).